Lecture 19: Ring of fractions, modules of fractions

R = commutative ring.

Last Time: We defined multiplicatively closed sets S & the ring of fractions $S^{-1}R$

Def: Fix a commutative ring R & $S \subset R$. We say S is multiplicatively closed if:

(i) $0 \notin S$ ⇝ otherwise localization gives a set with $0=1$.

(ii) $1 \in S$

(iii) $a, b \in S \Rightarrow ab \in S$.

- Equivalence relation $\sim$ on $R \times S$:

$$\boxed{(a,s) \sim (b,t) \iff \exists\, s' \in S \text{ with } s'(at-bs)=0}$$

Def The ring of fractions of R relative to S, denoted by $S^{-1}R$ is the set $R \times S/\sim$ with

1. Addition: $(a,s) + (b,t) = (at+bs, st)$
2. Multiplication: $(a,s) \cdot (b,t) = (ab, st)$
3. Neutral elements: $0 = (0,1)$ & $1 = (1,1)$

(Think of (a,s) in $R^{-1}S$ as $\frac{a}{s}$.)

Special case: R is an integral domain (no zero divisors)

Then $S = R \setminus \{0\}$ is a multiplicatively closed set.

Then $S^{-1}R$ is a field called the field of fractions

Sometimes denoted by $\mathrm{Quot}(R)$

$\boxed{\text{More examples}}$

① Fix R commutative ring $\rightsquigarrow$ $\boxed{S = \text{set of nonzero divisors of } R}$

Thm: S is multiplicatively closed

<u>Def</u>: $S^{-1}R$ = Total ring of fractions = $\boxed{\mathrm{Quot}(R)}$

② $S = \{1, x^i : i \geq 1\} \in \mathbb{K}[x]$ is multiplicatively closed.

Thm: $S^{-1}\mathbb{K}[x] =$

Universal Properties

Proposition: We have a natural ring homomorphism $j_S : R \longrightarrow S^{-1}R$, $a \longmapsto \frac{a}{1}$

such that for every $t \in S$, $j_S(t)$ is invertible in $S^{-1}R$ $\left(j_S(t)^{-1} = \frac{1}{t} \right)$.

Lemma: $\operatorname{Ker}(j_S)$

Theorem: Fix B another commutative ring & let $f : R \longrightarrow B$ be a ring homomorphism such that $\forall t \in S$: $f(t) \in B$ is invertible.
Then, there exists a unique ring homomorphism $\tilde{f} : S^{-1}R \longrightarrow B$ making $\tilde{f} \circ j_S = f$ ie:

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "j_S"'] & B \\ S^{-1}R \arrow[ur, dashed, "\exists! \tilde{f}"'] \arrow[u, phantom, shift left=8, "\circlearrowleft"] & \end{tikzcd}$$

Proof Want to show

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "j_S"'] \arrow[dr, phantom, "\circlearrowleft"] & B \\ S^{-1}R \arrow[ur, dashed, "\exists!\, \tilde{f}"'] & \end{tikzcd}$$

=   s

Modules of fractions

R comm. ring
$S \subset R$ mult. closed.

Given M an R-module, we want to define a "module of fractions".
R is an R-module $\Rightarrow$ mimic the construction of $S^{-1}R = R \times S/\sim$.

- Equivalence Relation on $M \times S$:

  $(m,s) \sim (m',s')$ if

Def: $S^{-1}M := M \times S/\sim$ = module of fractions of M relative to S [Write $\overline{(m,s)} = \frac{m}{s}$]

We make this into an abelian group:

- Addition: $\frac{m}{s} + \frac{m'}{s'} =$
- Neutral element: $\frac{0}{1}$

Proposition: The abelian group $S^{-1}M$ is both an R-module & an $S^{-1}R$-module via $\frac{a}{s} \cdot \frac{m}{t} = \frac{a \cdot m}{st}$ $\forall a \in R$, $m \in M$, $s,t \in S$ (R-module via j_S: $a \cdot \frac{m}{t} = \frac{a}{1} \cdot \frac{m}{t}$)

Universal Properties

Proposition: We have a natural R-linear gp hom
$$i_S : M \longrightarrow S^{-1}M =: \tilde{M}$$
$$m \longmapsto \frac{m}{1}$$

Proposition: For each $s \in S$, consider the gp hom
$$\rho_{(s)} : \tilde{M} \longrightarrow \tilde{M}$$
$$m \longmapsto s \cdot m$$

We have $\rho_{(s)} \in \mathrm{End}_R(\tilde{M})$ & $\rho_{(s)}$ is invertible $\rho_{(s)}^{-1} = \rho\left(\frac{1}{s}\right) \in \mathrm{End}_R(\tilde{M})$

Theorem: Let N be another module over R such that $\forall s \in S$
$$\rho_{(s)} : N \longrightarrow N$$
$$n \longmapsto sn$$
is invertible (as $\rho_{(s)} \in \mathrm{End}_R(N)$)

Given an R-linear map $f : M \longrightarrow N$ there exists a unique R-linear gp hom $\tilde{f} : \tilde{M} \longrightarrow N$ st

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "i_S"'] & N \\
\tilde{M} \arrow[ur, dashed, "\exists ! \tilde{f}"'] \arrow[ur, phantom, bend left, "\circlearrowright"] &
\end{tikzcd}$$

commutes.

Proof:

$\boxed{\text{Ideals of } S^{-1}R}$

Fix R commutative ring, $S \subset R$ mult. closed & $\boxed{j_S : R \to S^{-1}R}$

given $\mathfrak{a} \subset R$ ideal, we define $\boxed{S^{-1}\mathfrak{a}}$ = ideal in $S^{-1}R$ generated by $j_S(\mathfrak{a})$.

Proposition: $S^{-1}\mathfrak{a}$ agrees with the module of fractions of $\mathfrak{a}$ (viewed as an R-module) relative to S.

Theorem: Every ideal in $S^{-1}R$ is of this form ($= S^{-1}\mathfrak{a}$ for some $\mathfrak{a} \subset R$ ideal)

Furthermore, $S^{-1}\mathfrak{a} = S^{-1}R \iff S \cap \mathfrak{a} \neq \emptyset$.

Last claim:

$$\boxed{S^{-1}\mathfrak{a} = S^{-1}R \iff S \cap \mathfrak{a} \neq \emptyset.}$$

Proposition: Prime ideals in $S^{-1}R$ are of the form $S^{-1}\mathfrak{p}$, where $\mathfrak{p} \subsetneq R$ is a prime ideal with $\mathfrak{p} \cap S = \emptyset$.