Lecture 19: Ring of fractions, modules of fractions

R = commutative ring.

Last Time: We defined multiplicatively closed sets S & the ring of fractions $S^{-1}R$

Def: Fix a commutative ring R & $S \subset R$. We say S is multiplicatively closed if:

(i) $0 \notin S$ $\rightsquigarrow$ otherwise localization gives a set with $0=1$.

(ii) $1 \in S$

(iii) $a, b \in S \Rightarrow ab \in S$.

- Equivalence relation $\sim$ on $R \times S$:

$$\boxed{(a,s) \sim (b,t) \iff \exists\, s' \in S \text{ with } s'(at-bs)=0}$$

Def The ring of fractions of R relative to S, denoted by $S^{-1}R$ is the set $R \times S/\sim$ with

1. Addition: $(a,s) + (b,t) = (at+bs, st)$
2. Multiplication: $(a,s)\cdot(b,t) = (ab, st)$
3. Neutral elements: $0 = (0,1)$ & $1 = (1,1)$

(Think of (a,s) in $R^{-1}S$ as $\frac{a}{s}$.)

Special case: R is an integral domain (no zero divisors)

Then $S = R \setminus \{0\}$ is a multiplicatively closed set.

Then $S^{-1}R$ is a field called the field of fractions

Sometimes denoted by $\mathrm{Quot}(R)$

$(a,s) \neq (0,1)$ is invertible & $(a,s)^{-1} = (s,a)$.

$(\Leftrightarrow a \neq 0)$

Examples ① $R = \mathbb{Z}$, $\mathrm{Quot}(R) = \mathbb{Q}$ $\quad (a,b) \longleftrightarrow \frac{a}{b}$

$(a,b) \sim (c,d) \iff \exists\, s \in \mathbb{Z} \setminus \{0\}$ with $s(ad - bc) = 0$

But $\mathbb{Z}$ is a domain, so $ad - bc = 0$ (since $s \neq 0$)

If $m = \gcd(a,b)$, $n = \gcd(d,c)$ then $\frac{a}{m}\frac{d}{n} = \frac{b}{m}\frac{c}{n}$ forces $\frac{a}{m} = k\frac{c}{n}$

& $\frac{d}{n} = k\frac{b}{m}$ for $k \in \mathbb{Z}$ $\Rightarrow \frac{a}{b} = \frac{a/m}{b/m} = \frac{c/n}{d/n} = \frac{c}{d}$.

② $R = \mathbb{Z}[x]$, $\mathrm{Quot}(R) = \mathbb{Q}(x) = \left\{ \frac{P(x)}{Q(x)} : P, Q \in \mathbb{Q}[x],\ Q \neq 0 \right\}$

More examples

1. Fix R commutative ring $\rightsquigarrow$ $\boxed{S = \text{set of nonzero divisors of } R}$

Thm: S is multiplicatively closed

<u>Def</u>: $S^{-1}R$ = Total ring of fractions = $\boxed{\mathrm{Quot}(R)}$

2. $S = \{1, x^i : i \geq 1\} \in \mathbb{K}[x]$ is multiplicatively closed.

Thm: $S^{-1}\mathbb{K}[x] = \mathbb{K}[x, x^{-1}]$. (Laurent polynomials / $\mathbb{K}$)

$(a,s) \sim (b,t) \iff x^n(at - bs) = 0 \quad n \geq 0$

Again, $at - bs = 0$, $s, t \in S$ so $s = x^k, t = x^l$

So
$$\left.\begin{array}{ll} a = b\,x^{k-l} \in \mathbb{K}[x] & \text{if } k \geq l \\ b = a\,x^{l-k} \in \mathbb{K}[x] & \text{if } l \geq k \end{array}\right\} \Longrightarrow \frac{a}{s} \in \mathbb{K}[x, x^{-1}].$$

3. $R = \mathbb{Z}/6\mathbb{Z}$ $S = \{1, 2, 4\}$ $\rightsquigarrow$ $\forall s \in S$: $\frac{3}{s} = 0$ since $2(3 \cdot 1 - 0 \cdot s) = 0$ in R

<u>Claim</u>: $S^{-1}R = \{0, 1, \frac{1}{2}\}$ (HW7)

Universal Properties

Proposition: We have a natural ring homomorphism
$$j_S : \begin{array}{ccc} R & \longrightarrow & S^{-1}R \\ a & \longmapsto & \frac{a}{1} \end{array}$$
such that for every $t \in S$, $j_S(t)$ is invertible in $S^{-1}R$ $\left(j_S(t)^{-1} = \frac{1}{t} \right)$

Proof: Definition of the ring structure on $S^{-1}R$ makes j_S a ring homomorphism

- $\left(\frac{t}{1}\right)^{-1} = \frac{1}{t}$ because $(t,1)\cdot(1,t) = (t,t) = (1,1)$

Lemma: $\operatorname{Ker}(j_S) = \{ a \in R : \exists s \in S \text{ with } sa = 0 \}$ $\left(\frac{a}{1} = \frac{0}{1} \iff sa = 0 \text{ for some } s \in R \right)$

Theorem: Fix B another commutative ring & let $f : R \longrightarrow B$ be a ring homomorphism such that $\forall t \in S$: $f(t) \in B$ is invertible. Then, there exists a unique ring homomorphism $\tilde{f} : S^{-1}R \longrightarrow B$ making $\tilde{f} \circ j_S = f$ i.e.:

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "j_S"'] & B \\ S^{-1}R \arrow[ur, dashed, "\exists ! \tilde{f}"'] \arrow[u, phantom, ""] & \end{tikzcd}$$

Proof Want to show

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "j_S"'] \arrow[dr, phantom, "\circlearrowleft"] & B \\ S^{-1}R \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

s.t. $\boxed{\tilde{f}\left(\frac{a}{s}\right) := f(a)\, f(s)^{-1}}$

ok since $f(s) \in B^{\times}$.

- Well-defined: $\frac{a}{s} = \frac{b}{t} \iff \exists\, s' \in S$ with $s'(at - bs) = 0$

Then $\underset{B^{\times}}{\underset{\cap}{f(s')}}\,(f(a)f(t) - f(b)f(s)) = 0. \implies f(a)f(t) = f(b)f(s)$

So $f(a)\, f(s)^{-1} = f(b)\, f(t)^{-1}$ in B

- $\tilde{f} \circ j_S(a) = \tilde{f}\left(\frac{a}{1}\right) = f(a)\, f(1)^{-1} = f(a)\, 1^{-1} = f(a)$.

- Ring homomorphism:

- $\tilde{f}\left(\frac{a}{s} + \frac{b}{t}\right) = \tilde{f}\left(\frac{at+bs}{st}\right) = f(at+bs)\, f(st)^{-1} = (f(a)f(t) + f(b)f(s))\, f(s)^{-1} f(t)^{-1}$
$= f(a)\, f(s)^{-1} + f(b)\, f(t)^{-1} = \tilde{f}\left(\frac{a}{s}\right) + \tilde{f}\left(\frac{b}{t}\right)$

- $\tilde{f}(1) = \tilde{f}\left(\frac{1}{1}\right) = f(1)\, f(1)^{-1} = 1 \cdot 1^{-1} = 1$.

- $\tilde{f}\left(\frac{a}{s}\frac{b}{t}\right) = \tilde{f}\left(\frac{ab}{st}\right) = f(ab)\, f(st)^{-1} = f(a) f(b)\, f(s)^{-1} f(t)^{-1} = f(a) f(s)^{-1} f(b) f(t)^{-1} =$
$= \tilde{f}\left(\frac{a}{s}\right) \tilde{f}\left(\frac{b}{t}\right)$.

Modules of fractions

R comm. ring
$S \subset R$ mult. closed.

Given M an R-module, we want to define a "module of fractions".
R is an R-module $\Rightarrow$ mimic the construction of $S^{-1}R = R \times S/\sim$.

- Equivalence Relation on $M \times S$:

$(m,s) \sim (m',s')$ if there exists $t \in S$ such that: $\underset{\in R}{t}(\underbrace{s'm - sm'}_{\text{in } M}) = 0$ in M.

<u>Exercise</u>: This is an equivalence relation (HW7)

<u>Def</u>: $S^{-1}M := M \times S/\sim$ = module of fractions of M relative to S [Write $\overline{(m,s)} = \frac{m}{s}$]

We make this into an abelian group:

- <u>Addition</u>: $\frac{m}{s} + \frac{m'}{s'} = \frac{s' m + s \cdot m'}{ss'}$
- <u>Neutral element</u>: $\frac{0}{1}$

<u>Proposition</u>: The abelian group $S^{-1}M$ is both an R-module & an $S^{-1}R$-module via $\frac{a}{s} \cdot \frac{m}{t} = \frac{a \cdot m}{st}$ $\forall a \in R$, $m \in M$, $s,t \in S$ (R-module via j_S: $a \cdot \frac{m}{t} = \frac{a}{1} \cdot \frac{m}{t}$)

Universal Properties

Proposition: We have a natural R-linear gp hom $i_S : M \longrightarrow S^{-1}M =: \tilde{M}$

$$m \longmapsto \frac{m}{1}$$

Proposition: For each $s \in S$, consider the gp hom $\rho_{(s)} : \tilde{M} \longrightarrow \tilde{M}$

$$m \longmapsto s \cdot m$$

We have $\rho_{(s)} \in \mathrm{End}_R(\tilde{M})$ & $\rho_{(s)}$ is invertible $\rho_{(s)}^{-1} = \rho(\frac{1}{s}) \in \mathrm{End}_R(\tilde{M})$

Theorem: Let N be another module over R such that $\forall s \in S$

$$\rho_{(s)} : N \longrightarrow N \quad \text{is invertible (as } \rho_{(s)} \in \mathrm{End}_R(N)\text{)}$$
$$n \longmapsto sn$$

Given an R-linear map $f : M \longrightarrow N$ there exists a unique R-linear gp hom $\tilde{f} : \tilde{M} \longrightarrow N$ st

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "i_S"'] & N \\
\tilde{M} \arrow[ur, dashed, "\exists ! \tilde{f}"'] \arrow[ur, phantom, bend left, "\circlearrowright"] &
\end{tikzcd}$$

commutes.

Proof: $\tilde{f}\left(\frac{m}{s}\right) = \rho_{(s)}^{-1}(f(m))$ Well defined, gp hom & R-linear (exercise)

$\tilde{f} \circ i_S(m) = \rho(1)^{-1}(f(m)) = id^{-1}(f(m)) = f(m)$. $\square$

$\boxed{\text{Ideals of } S^{-1}R}$

Fix R commutative ring, $S \subset R$ mult. closed & $\boxed{j_S : R \to S^{-1}R}$

given $\mathfrak{a} \subset R$ ideal, we define $\boxed{S^{-1}\mathfrak{a}}$ = ideal in $S^{-1}R$ generated by $j_S(\mathfrak{a})$.

Proposition: $S^{-1}\mathfrak{a}$ agrees with the module of fractions of $\mathfrak{a}$ (viewed as an R-module) relative to S.

Theorem: Every ideal in $S^{-1}R$ is of this form ($= S^{-1}\mathfrak{a}$ for some $\mathfrak{a} \subset R$ ideal)

Furthermore, $S^{-1}\mathfrak{a} = S^{-1}R \iff S \cap \mathfrak{a} \neq \emptyset$.

Pf/ Let $\mathfrak{b} \subset S^{-1}R$ be an ideal. & set $\mathfrak{a} = j_S^{-1}(\mathfrak{b})$. (ideal because j_S is ring hom)

• **Claim:** $S^{-1}\mathfrak{a} = \mathfrak{b}$

($\subseteq$) $a \in \mathfrak{a}$, $\frac{r}{s} \in S^{-1}R \Rightarrow \frac{r}{s}\cdot\frac{a}{1} = \underbrace{\boxed{\frac{r}{s}}}_{\in S^{-1}R}\ \overbrace{\boxed{\frac{a}{1}}}^{\in \mathfrak{b}} \in \mathfrak{b}$ (ideal) So $S^{-1}\mathfrak{a} = S^{-1}R\left(\frac{a}{1} : a \in \mathfrak{a}\right) \subseteq \mathfrak{b}$

($\supseteq$) Pick $x \in \mathfrak{b}$. Then $x = \frac{y}{s}$ for some $y \in R$, $s \in S$ $\Rightarrow \frac{y}{1} = \underbrace{\boxed{\frac{s}{1}}}_{\in R} \cdot \underbrace{\boxed{\frac{y}{s}}}_{\in \mathfrak{b}}$ so $y \in \mathfrak{a}$ & $x \in S^{-1}\mathfrak{a}$

This shows: $\mathfrak{b} \subseteq S^{-1}\mathfrak{a}$.

<u>Last claim</u>: $\boxed{S^{-1}\mathfrak{a} = S^{-1}R \iff S\cap\mathfrak{a} \neq \emptyset.}$

($\Leftarrow$) If $s\in S\cap\mathfrak{a}$, write $1 = \frac{s}{s} \in S^{-1}\mathfrak{a}$.

($\Rightarrow$) Assume $1\in S^{-1}\mathfrak{a}$. Then, $\exists\ a_1,\dots,a_n\in\mathfrak{a}$, $r_1,\dots,r_n\in R$

& $s_1,\dots,s_n\in S$ with $1 = \sum_{i=1}^{n} \frac{r_i}{s_i}\cdot a_i = \sum_{i=1}^{n} \frac{(r_i a_i)}{s_i} = \sum_{i=1}^{n} \frac{b_i}{s}$

where $s = \prod_{i=1}^{n} s_i$ & $b_i = r_i a_i \prod_{j\neq i} s_j \in \mathfrak{a}$ $\forall i=1,\dots,n$

BUT $\sum_{i=1}^{n} \frac{b_i}{s} = \frac{\sum_{i=1}^{n} b_i}{s} = \frac{b}{s}$ with $b = \sum_{i=1}^{n} b_i \in \mathfrak{a}$

So $1 = \frac{b}{s}$ for $b\in\mathfrak{a}$, $s\in S$

<u>Conclude</u>: $\exists\ t\in S$ with $0 = t(s\cdot 1 - b\cdot 1) = ts - tb$

So $\underset{S}{\underset{\cap}{ts}} = \underset{\mathfrak{a}}{\underset{\cap}{tb}} \in S\cap\mathfrak{a}$.

$\square$

Proposition: Prime ideals in $S^{-1}R$ are of the form $S^{-1}\mathfrak{p}$, where $\mathfrak{p} \subsetneq R$ is a prime ideal with $\mathfrak{p} \cap S = \emptyset$.

Proof Let $\mathfrak{q} \subsetneq S^{-1}R$ be a prime ideal. By the proof of the previous Theorem, we know $\mathfrak{q} = S^{-1}\mathfrak{a}$ where $\mathfrak{a} = j_S^{-1}(\mathfrak{q})$.

Since $\mathfrak{q}$ is prime & j_S is a ring homomorphism, we know $\mathfrak{a}$ is a prime ideal of R. But $\mathfrak{q} \subsetneq S^{-1}R \Rightarrow \mathfrak{a} \cap S = \emptyset$

Conversely, given $\mathfrak{p} \subsetneq R$ prime with $\mathfrak{p} \cap S = \emptyset$, we want to show $S^{-1}\mathfrak{p} \subseteq S^{-1}R$ is a prime ideal.

- Properness follows since $\mathfrak{p} \cap S = \emptyset$
- $S^{-1}\mathfrak{p}$ is an ideal of $S^{-1}R$ by the Theorem.
- $\frac{a}{s} \cdot \frac{b}{t} \in S^{-1}\mathfrak{p}$ with $a, b \in R$ $s, t \in R$ $\Rightarrow \frac{ab}{1} = \frac{(st)}{1} \frac{ab}{st} \in S^{-1}\mathfrak{p}$

$\Rightarrow ab \in \mathfrak{p} \underset{\mathfrak{p}\text{ prime}}{\Longrightarrow} a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. $\Rightarrow \frac{a}{s} \in S^{-1}\mathfrak{p}$ or $\frac{b}{t} \in S^{-1}\mathfrak{p}$. $\square$