Lecture 20: Localization & Noetherian rings

Recall: Rings & modules of fractions for R commutative ring

$S \subset R$ mult. closed set $\rightsquigarrow$ $S^{-1}R$ = another comm ring

M : R-module $\rightsquigarrow$ $S^{-1}M$: an $S^{-1}R$-module.

Prop: (1) Every ideal of $S^{-1}R$ is of the form $S^{-1}\mathfrak{a}$ for $\mathfrak{a} \subset R$ ideal

& $S^{-1}\mathfrak{a} = S^{-1}R \iff S \cap \mathfrak{a} \neq \emptyset$.

(2) Prime ideals of $S^{-1}R \xleftrightarrow{\text{1-to-1}}$ prime ideals of R not meeting S

$(j_S(\mathfrak{p}))S^{-1}R = S^{-1}\mathfrak{p} \longleftarrow\!\shortmid$ $\mathfrak{p}$

TODAY: Localization & first properties of Noetherian rings

GOAL: Build suitable S where $S^{-1}R$ is a **local ring** (unique max l ideal)

Geometrically: Study a space X in the neighborhood of a point.

$R = \{$polynomial functions $X \longrightarrow \mathbb{C}\}$.

Localization

Fix $\mathfrak{p} \subsetneq R$ a prime ideal and let $\boxed{S = R \setminus \mathfrak{p}}$.

Lemma: S is multiplicatively closed.

Def: $R_{\mathfrak{p}} := S^{-1}R$ is called the localization of R at the prime ideal $\mathfrak{p}$.

Proposition: $R_{\mathfrak{p}}$ is a local ring with unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$.

Obs: If R is a domain, $j_S : R \hookrightarrow \mathrm{Quot}(R) = (R \setminus \{0\})^{-1}R$

So $\boxed{R \hookrightarrow R_{\mathfrak{p}} \hookrightarrow \mathrm{Quot}(R)}$

$\boxed{\text{Examples}}$

$R \supseteq \mathfrak{p}$ prime ideal

$S = R \setminus \mathfrak{p}$

① $R = \mathbb{Z}$

② $R = \mathbb{C}[x]$

③ $R = \mathbb{C}[x, y]$

Localization for R-modules

Def: Given M an R-module, we define its <u>localization at $\mathfrak{p}$</u> as $\boxed{M_{\mathfrak{p}} = S^{-1}M}$ where $S = R \setminus \mathfrak{p}$.

Q: What is $\ker\left(M \xrightarrow{i_S} S^{-1}M \right)$?

A:

Def: $\mathrm{Ann}(m) = \{ r \in R : rm = 0 \}$ (Annihilator of m)

Lemma: $\mathrm{Ann}(m)$ is an ideal of R. It is proper $\Leftrightarrow$ $m \neq 0$.

Localizations are useful tools to decide when modules are trivial, ie

Theorem: $M = \{0\} \iff M_{\mathfrak{p}} = 0 \quad \forall \, \mathfrak{p} \subsetneq R$ prime ideal

$\iff M_{\mathfrak{m}} = 0 \quad \forall \, \mathfrak{m} \subsetneq R$ mxl ideal

Proof: Want to show TFAE:

(1) $M = \{0\}$

(2) $M_{\mathfrak{p}} = 0 \quad \forall \mathfrak{p} \subsetneq R$ prime ideal

(3) $M_{\mathfrak{m}} = 0 \quad \forall \mathfrak{m} \subsetneq R$ mxl ideal

Theorem: Assume R is an integral domain. Then:

$$R = \bigcap_{\substack{\mathfrak{m} \text{ mxl} \\ \text{ideal}}} R_{\mathfrak{m}} = \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ \text{ideal}}} R_{\mathfrak{p}}$$

Modules of fractions and their homomorphisms

Fix R commutative ring & $S \subset R$ mult closed set.

Let M, N be two R-modules & set $f: M \longrightarrow N$ R-linear

Then $S^{-1}f: S^{-1}M \longrightarrow S^{-1}N$

Proposition: Let $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ be a ses of R-linear maps between R-modules. Then, the following sequence of $S^{-1}R$-linear maps is exact:

$$0 \longrightarrow S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M_2 \xrightarrow{S^{-1}g} S^{-1}M_3 \longrightarrow 0.$$

Obs: Can use this to give an alternative proof of Thm on previous slide.

Proof: To show $\boxed{0 \longrightarrow S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M_2 \xrightarrow{S^{-1}g} S^{-1}M_3 \longrightarrow 0 \text{. is ses}}$

<u>Corollary</u>: (1) Let $N \subset M$ be submodule over R.

Then $\dfrac{S^{-1}M}{S^{-1}N}$ (as $S^{-1}R$ - modules)

(2) In particular, for an ideal $\mathfrak{a} \subset R$, we have

$S^{-1}R / S^{-1}\mathfrak{a}$

Noetherianness - First properties

<u>Definition</u>: A commutative ring R is called <u>Noetherian</u> if for every chain of ideals of R $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \dots$ there is $k \geq 0$ with $\mathfrak{a}_k = \mathfrak{a}_{k+1} = \dots$

<u>Theorem</u>: The following conditions on a commutative ring R are equivalent:

(1) R is Noetherian

(2) Every nonempty set $\mathcal{S}$ of ideals of R has a maximal element

(3) Every ideal $\mathfrak{a} \subseteq R$ is finitely generated.

ACC $\Longleftrightarrow$ every collection $\neq \emptyset$ of ideals has a max. element $\Longleftrightarrow$ every ideal is f.g.