Lecture 20: Localization & Noetherian rings

Recall: Rings & modules of fractions for R commutative ring

$S \subset R$ mult. closed set $\rightsquigarrow$ $S^{-1}R$ = another comm ring

M : R-module $\rightsquigarrow$ $S^{-1}M$: an $S^{-1}R$-module.

Prop: (1) Every ideal of $S^{-1}R$ is of the form $S^{-1}\mathfrak{a}$ for $\mathfrak{a} \subset R$ ideal

& $S^{-1}\mathfrak{a} = S^{-1}R \iff S \cap \mathfrak{a} \neq \emptyset$.

(2) Prime ideals of $S^{-1}R$ $\xleftrightarrow{1\text{-to-}1}$ prime ideals of R not meeting S

$(j_S(\mathfrak{p}))S^{-1}R = S^{-1}\mathfrak{p} \longleftarrow\!\!\shortmid$ $\mathfrak{p}$

TODAY: Localization & first properties of Noetherian rings

GOAL: Build suitable S where $S^{-1}R$ is a local ring (unique max'l ideal)

Geometrically: Study a space X in the neighborhood of a point.

$R = \{$polynomial functions $X \longrightarrow \mathbb{C}\}$.

$\boxed{\text{Localization}}$

Fix $\mathfrak{p} \subsetneq R$ a prime ideal and let $\boxed{S = R \setminus \mathfrak{p}}$.

__Lemma__: S is multiplicatively closed.

Pf/ • $1 \in S$ $(1 \notin \mathfrak{p})$ & • $0 \notin S$ $(0 \in \mathfrak{p})$

• $a, b \in S$ means $a, b \notin \mathfrak{p}$ so $ab \notin \mathfrak{p}$ because $\mathfrak{p}$ is prime $\Rightarrow ab \in S$. □

__Def__: $R_{\mathfrak{p}} := S^{-1}R$ is called the localization of R at the prime ideal $\mathfrak{p}$.

__Proposition__: $R_{\mathfrak{p}}$ is a local ring with unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$.

__Proof__: Let $\mathfrak{b}$ be a proper ideal of $R_{\mathfrak{p}}$. By Prop (1) $\mathfrak{b} = S^{-1}\mathfrak{a}$ for $\mathfrak{a} \subset R$ ideal. If $\mathfrak{a} \cap S \neq \emptyset$, then $\mathfrak{b} = S^{-1}R$. __Contr!__

So $\mathfrak{a} \cap S = \emptyset$, meaning $\mathfrak{a} \subset \mathfrak{p}$. Hence $\mathfrak{b} \subseteq \mathfrak{p}(S^{-1}R)$.

So every proper ideal of $R_{\mathfrak{p}}$ lies in $\mathfrak{p}R_{\mathfrak{p}}$. Thus $(R_{\mathfrak{p}}, \mathfrak{p}_{\mathfrak{p}})$ is local □

__Obs__: If R is a __domain__, $j_S : R \longrightarrow \mathrm{Quot}(R) = (R \setminus \{0\})^{-1}R$

So $\boxed{R \hookrightarrow R_{\mathfrak{p}} \hookrightarrow \mathrm{Quot}(R)}$

$\boxed{\text{Examples}}$

$R \supsetneq \wp$ prime ideal
$S = R \setminus \wp$

1. $R = \mathbb{Z}$ $\quad (p)$ is prime ideal $\rightsquigarrow S = \{ b \in \mathbb{Z} : p \nmid b \}$

$$\rightsquigarrow \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} : \ a, b \in \mathbb{Z} \ \ \gcd(a,b) = 1, \ \ p \nmid b \right\}$$

2. $R = \mathbb{C}[x]$ $\quad (x)$ is mxl ideal, so prime

$$R_{(x)} = \left\{ \frac{P}{Q} : \ P, Q \in \mathbb{C}[x] \ : \ x \nmid Q \right\}$$

3. $R = \mathbb{C}[x, y]$ $\quad \mathfrak{m} = (x, y)$ is mxl ideal, so prime

$$R_{(x,y)} = \left\{ \frac{P}{Q} : \ P, Q \in \mathbb{C}[x,y] \ \ Q(0,0) \neq 0 \right\}$$

$$\boxed{\text{Localization for } R\text{-modules}}$$

Def: Given M an R-module, we define its localization at $\mathfrak{p}$ as $\boxed{M_{\mathfrak{p}} = S^{-1}M}$ where $S = R \setminus \mathfrak{p}$.

Q: What is $\ker\left(M \xrightarrow{i_S} S^{-1}M \right)$?

A: $\ker(i_S) = \{ m \in M : \exists\, s \in S \text{ with } s \cdot m = 0 \text{ in } M \}$

Def: $\mathrm{Ann}(m) = \{ r \in R : rm = 0 \}$ (Annihilator of m)

Lemma: $\mathrm{Ann}(m)$ is an ideal of R. It is proper $\Leftrightarrow$ $m \neq 0$.

Consequence: $m \in \ker(i_S) \Leftrightarrow \mathrm{Ann}(m) \cap S \neq \emptyset$.

Localizations are useful tools to decide when modules are trivial, ie

Theorem:

$$M = \{0\} \iff M_{\mathfrak{p}} = 0 \quad \forall\, \mathfrak{p} \subsetneq R \text{ prime ideal}$$
$$\iff M_{\mathfrak{m}} = 0 \quad \forall\, \mathfrak{m} \subsetneq R \text{ mxl ideal}$$

Proof: Want to show TFAE:

(1) $M = \{0\}$

(2) $M_{\mathfrak{p}} = 0 \quad \forall\, \mathfrak{p} \subsetneq R$ prime ideal

(3) $M_{\mathfrak{m}} = 0 \quad \forall\, \mathfrak{m} \subsetneq R$ maxl ideal

Proof: (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) is clear (maxl ideals are prime)

To finish, we prove (3) $\Longrightarrow$ (1): We argue by contradiction.

Pick $m \in M \setminus \{0\}$ & let $\mathfrak{a} = \mathrm{Ann}(m) \subsetneq R$. Pick $\mathfrak{m} \subset R$ maximal ideal with $\mathfrak{a} \subset \mathfrak{m}$. By hypotheses $M_{\mathfrak{m}} = 0$, so

$\frac{m}{1} = 0$ in $M_{\mathfrak{m}}$ meaning $\exists\, s \in R \setminus \mathfrak{m}$ with $sm = 0$. This

cannot happen since $(R \setminus \mathfrak{m}) \cap \mathrm{Ann}(m) = \emptyset$. $\square$

Theorem: Assume R is an integral domain. Then:

$$R = \bigcap_{\substack{\mathfrak{m} \text{ mxl} \\ \text{ideal}}} R_{\mathfrak{m}} = \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ \text{ideal}}} R_{\mathfrak{p}}$$

Proof We know $R \hookrightarrow R_{\mathfrak{m}} \hookrightarrow \operatorname{Quot}(R)$ $\forall \mathfrak{m}$

Write $\tilde{R} = \bigcap_{\mathfrak{m} \text{ mxl}} R_{\mathfrak{m}} \supseteq R$. We view $\tilde{R}/R$ as an R-module

To show: $\tilde{R}/R = 0$. Pick $x \in \tilde{R}$ & write $x = \frac{a}{b}$ $\begin{matrix} a \in R \\ b \in R \end{matrix}$

To show $a \in (b)R$ so $x \in R$.

Consider $I = \{ t \in R : t\frac{a}{b} \in R \} = \operatorname{Ann}(\frac{a}{b})$

This means $\frac{ta}{b} = \frac{a'}{1}$ for $a' \in R$, ie $ta = a'b$

Thus $I = \{ t \in R : ta \in R(b) \}$.

If $I = (1)$ we are done $1 \cdot a = a \in R(b)$. Else, $\exists\, \mathfrak{m} \subsetneq R$ mxl with $I \subseteq \mathfrak{m}$. Since $\frac{a}{b} \in \tilde{R} \subseteq R_{\mathfrak{m}}$ so $\frac{a}{b} = \frac{a'}{b'}$, with $b' \notin \mathfrak{m}$.

$\Rightarrow b'a = a'b$ so $b' \in I \subseteq \mathfrak{m}$ Contr!. $\square$

Modules of fractions and their homomorphisms

Fix R commutative ring & $S \subset R$ mult closed set.

Let M, N be two R-modules & set $f : M \longrightarrow N$ R-linear

Then $S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$

$$\frac{m}{s} \longmapsto \frac{f(m)}{s}$$

Proposition: Let $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ be a ses of R-linear maps between R-modules. Then, the following sequence of $S^{-1}R$-linear maps is exact:

$$0 \longrightarrow S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M_2 \xrightarrow{S^{-1}g} S^{-1}M_3 \longrightarrow 0.$$

Obs: Can use this to give an alternative proof of Thm on previous slide.

Proof: To show $\boxed{0 \longrightarrow S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M_2 \xrightarrow{S^{-1}g} S^{-1}M_3 \longrightarrow 0 \text{. is ses}}$

(1) $\operatorname{Ker}(S^{-1}f) = \{\frac{m}{s} \in S^{-1}M_1 : \frac{f(m)}{s} = 0 \text{ in } S^{-1}M_2\}$

If $\frac{f(m)}{s} = 0$, then $f(m) = s \cdot \frac{f(m)}{s} = 0 \Rightarrow m \in \operatorname{Ker} f = \{0\}$
$m \in M$

Conclude: $\operatorname{Ker}(S^{-1}f) = \{0\}$.

(2) $S^{-1}g$ is surjective: Let $\frac{m_3}{s} \in S^{-1}M_3$, $m_3 \in M_3$, $s \in S$

Since g is surjective $\exists\, m_2 \in M_2$ st $g(m_2) = m_3$

So $\frac{m_3}{s} = \frac{g(m_2)}{s} = S^{-1}g\left(\frac{m_2}{s}\right)$. Conclude: $S^{-1}g$ is surjective.

(3) $\boxed{\operatorname{Ker}(S^{-1}g) = \operatorname{Im}(S^{-1}f)}$

($\supseteq$) $(S^{-1}g) \circ (S^{-1}f)\left(\frac{m_1}{s}\right) = S^{-1}(g)\left(\frac{f(m_1)}{s}\right) = \frac{g(f(m_1))}{s} = \frac{0}{s} = 0$ ✓

($\subseteq$) Conversely, if $\frac{m_2}{s} \in \operatorname{Ker}(S^{-1}g)$ then $\frac{g(m_2)}{s} = 0$ so

$g(m_2) = s\, \frac{g(m_2)}{s} = 0$ so $m_2 \in \operatorname{Ker} g = \operatorname{Im} f$ so

$m_2 = f(m_1)$ for some $m_1 \in M_1$. Thus, $\frac{m_2}{s} = f\left(\frac{m_1}{s}\right) \in \operatorname{Im} S^{-1}f$. $\square$

Corollary: (1) Let $N \subset M$ be submodule over R.

Then $\dfrac{S^{-1}M}{S^{-1}N} \simeq S^{-1}(M/N)$. (as $S^{-1}R$-modules)

(2) In particular, for an ideal $\mathfrak{a} \subset R$, we have

$$\frac{S^{-1}R}{S^{-1}\mathfrak{a}} \simeq S^{-1}(R/\mathfrak{a}) \underset{(*)}{\simeq} \bar{S}^{-1}(R/\mathfrak{a})$$ (as $S^{-1}R$-modules)

(*) if $S \cap \mathfrak{a} = \emptyset$
(otherwise we get $0 \in \bar{S}$.)

where $\bar{S}$ = image of S under $R \longrightarrow R/\mathfrak{a}$.

Pf. (1) Use $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ ses of R-mod

Then $0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0$ is ses of $S^{-1}R$-mod

(2) Need to show $S^{-1}(R/\mathfrak{a}) \simeq \bar{S}^{-1}(R/\mathfrak{a})$ $S^{-1}R$-mod iso

$$\frac{\bar{r}}{s} \longmapsto \frac{\bar{r}}{\bar{s}} \qquad \left(\frac{a}{b} \cdot \frac{\bar{r}}{s} = \frac{\overline{ar}}{bs}\right)$$

Well-def $\frac{\bar{r}}{s} = \frac{\bar{r'}}{s'} \iff \exists\, t \in S$ st $t(s'\bar{r} - s\bar{r'}) = 0$ in $R/\mathfrak{a}$.

$\iff t(s'r - sr') \in \mathfrak{a}$ & $t \in S$ $\iff \bar{t}(\bar{s'}\bar{r} - \bar{s}\bar{r'}) = 0$ in $R/\mathfrak{a}$ & $\bar{t} \in \bar{S}$. $\square$

Northerianness - First properties

<u>Definition</u>: A commutative ring R is called <u>Northerian</u> if for every chain of ideals of R $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \ldots$ there is $k \geq 0$ with $\mathfrak{a}_k = \mathfrak{a}_{k+1} = \ldots$ (it stabilizes)

[ACC = Ascending chain condition]

<u>Theorem</u>: The following conditions on a commutative ring R are equivalent:

(1) R is Northerian

(2) Every nonempty set $\mathcal{S}$ of ideals of R has a maximal element

(3) Every ideal $\mathfrak{a} \subseteq R$ is finitely generated.

ACC $\Longleftrightarrow$ every collection $\mathcal{S}$ of ideals has a max. element $\Longleftrightarrow$ every ideal is f.g.
$\neq \emptyset$

Proof: $\boxed{(1) \Longrightarrow (2)}$ Let $\mathfrak{a}_0 \in \mathcal{S}$. If $\mathfrak{a}_0$ is not maximal, $\exists$ $\mathfrak{a}_1 \in \mathcal{S}$ with $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1$. Continuing in this fashion, we get an ascending chain of ideals $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \cdots$ That doesn't stabilize. Contr! Then $\exists\, \mathfrak{a}_k$ maximal element of $\mathcal{S}$.

$\boxed{(2) \Longrightarrow (3)}$ Let $\mathfrak{a}$ be an ideal. Consider the set

$$\mathcal{S} = \{ \mathfrak{a}' \subseteq \mathfrak{a} : \mathfrak{a}' \text{ is a fin. gen. ideal of } R \}$$

We order $\mathcal{S}$ by inclusion.

By (2), this set has a maximal element, say $\tilde{\mathfrak{a}}$.
If $\tilde{\mathfrak{a}} \subsetneq \mathfrak{a}$, pick $x \in \mathfrak{a} \setminus \tilde{\mathfrak{a}}$. Then $(\tilde{\mathfrak{a}}, x) \in \mathcal{S}$, contradicting the maximality of $\tilde{\mathfrak{a}} < (\tilde{\mathfrak{a}}, x)$. Hence $\tilde{\mathfrak{a}} = \mathfrak{a}$, which means $\mathfrak{a}$ is finitely generated since $\mathfrak{a} \in \mathcal{S}$.

$\boxed{(3) \Rightarrow (1)}$ Let $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \dots$ be a chain of ideals of R

Take $\mathfrak{a} = \bigcup_{i=0}^{\infty} \mathfrak{a}_i \subset R$.

By construction, $\mathfrak{a}$ is an ideal of R, thus finitely generated

by elements $a_1, \dots, a_n \in \mathfrak{a}$. Now, each $a_l \in \mathfrak{a}_{j_l}$ for some $j_l \geq 0$

Thus, $\mathfrak{a} = \mathfrak{a}_j$ for $j = \max\{j_1, j_2, \dots, j_n\}$ and so

$\mathfrak{a} = \mathfrak{a}_j = \mathfrak{a}_{j+1} = \dots$ The chain terminates, so A is Noetherian. $\square$