Lecture 21: Noetherian modules & Hilbert Basis Thm

Last Time: Saw 3 characterizations of Noetherian rings

Theorem: The following conditions on a commutative ring R are equivalent:

(1) R is Noetherian (ACC: Every ascending chain of ideals stabilizes)

(2) Every nonempty set $\mathcal{S}$ of ideals of R has a maximal element

(3) Every ideal $\mathfrak{a} \subseteq R$ is finitely generated.

Corollary: (1) Principal ideal domains are Noetherian (eg $\mathbb{Z}$, $\mathbb{Q}[x]$)

(2) If $f: A \longrightarrow B$ is a ring homomorphism with A, B commutative

Assume f is surjective. If A is Noetherian, so is B.

(3) Rings of fractions of Noetherian rings are Noetherian. In particular, localizations preserve Noetherianness.

⚠ Subrings of Noetherian rings need not be Noetherian.

Main Theorem for Noetherian rings:

Hilbert Basis Thm: If A is Noetherian, so is $A[x]$.

Hence $\mathbb{Z}[x_1, \ldots, x_n]$, $\mathbb{K}[x_1, \ldots, x_n]$ are Noetherian for any field $\mathbb{K}$.

- To prove this result, we'll need the notion of Noetherian modules.

Noetherian modules

Fix R = commutative ring. & M an R-module.

<u>Def</u>: We say M is <u>Noetherian</u> if it satisfies the ascending chain condition for submodules:

We have the following analog of Theorem 1:

<u>Theorem 2</u>: Fix R a commutative ring & M an R-module. TFAE:

(1) M is Noetherian

(2) Every nonempty set $\mathcal{S}$ of submodules of M has a maximal element

(3) Every submodule of M is finitely generated.

The proof is exactly the same as that of Thm 1.

Corollary 2: Let $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ be a ses of R-modules. Then: M_2 is Noetherian if, and only if, M_1 & M_3 are.

Proposition: Let R be a Noetherian ring & M an R-module. Then M is Noetherian if & only if M is finitely generated, ie $\exists\ x_1, \ldots, x_\ell \in M$ st every x in M can be written (no necessarily uniquely) as $x = a_1 x_1 + \cdots + a_\ell x_\ell$ for $a_1, \ldots, a_\ell \in R$.

($\Leftarrow$) Assume $M = \langle x_1, \dots x_\ell \rangle$ is f.g. R-module. Want to show: M is Noeth

$\boxed{\text{Examples}}$

Hilbert Basis Theorem

Theorem 3: If R is commutative and Noetherian, so is $R[X]$.