Lecture 21: Noetherian modules & Hilbert Basis Thm

Last Time: Saw 3 characterizations of Noetherian rings

Theorem: The following conditions on a commutative ring R are equivalent:

(1) R is Noetherian (ACC: Every ascending chain of ideals stabilizes)

(2) Every nonempty set $\mathcal{S}$ of ideals of R has a maximal element

(3) Every ideal $\mathfrak{a} \subseteq R$ is finitely generated.

Corollary: (1) Principal ideal domains are Noetherian (eg $\mathbb{Z}$, $\mathbb{C}[x]$)

(2) If $f: A \longrightarrow B$ is a ring homomorphism with A, B commutative. Assume f is surjective. If A is Noetherian, so is B.

(3) Rings of fractions of Noetherian rings are Noetherian. In particular, localizations preserve Noetherianness.

Proof (2): If $\mathfrak{b} \subset B$ is an ideal, $\mathfrak{a} = f^{-1}(\mathfrak{b}) \subset A$ is an ideal, So $\mathfrak{a} = (a_1, \ldots, a_n)$. Then $\underset{\substack{\uparrow \\ f \text{ surj}}}{\mathfrak{b} = f(\mathfrak{a})} = (f(a_1), \ldots, f(a_n))$. $\square$

⚠ Subrings of Noetherian rings need not be Noetherian.

Ex: Take $R = \mathbb{C}[x_1, x_2, \dots] = \bigcup_{n \in \mathbb{N}} \mathbb{C}[x_1, x_2, \dots, x_n]$

R is not Noetherian since

$\mathfrak{a}_1 = (x_1) \subsetneq \mathfrak{a}_2 = (x_1, x_2) \subsetneq \dots$ never terminates.

But R is a domain & $R \hookrightarrow \mathrm{Quot}(R)$ = field.

Since the only ideals of $\mathrm{Quot}(R)$ are (0) & (1), we get that $\mathrm{Quot}(R)$ is Noetherian. $\square$

Main Theorem for Noetherian rings:

Hilbert Basis Thm: If A is Noetherian, so is $A[x]$.

Hence $\mathbb{Z}[x_1, \dots, x_n]$, $\mathbb{K}[x_1, \dots, x_n]$ are Noetherian for any field $\mathbb{K}$.

. To prove this result, we'll need the notion of Noetherian modules.

Noetherian modules

Fix R = commutative ring. & M an R-module.

Def: We say M is Noetherian if it satisfies the ascending chain condition for submodules:

"Every chain of submodules $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ stabilizes ie $\exists\, \ell \geq 0$ st $M_\ell = M_{\ell+1} = \cdots$"

We have the following analog of Theorem 1:

Theorem 2: Fix R a commutative ring & M an R-module. TFAE:

(1) M is Noetherian

(2) Every nonempty set $\mathcal{S}$ of submodules of M has a maximal element

(3) Every submodule of M is finitely generated.

The proof is exactly the same as that of Thm 1.

Obs: R is a Noetherian ring if, and only if, it is a Noetherian R-module.

Corollary 2: Let $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ be a ses of R-modules. Then: M_2 is Noetherian if, and only if, M_1 & M_3 are.

Proof: $\Rightarrow$) submodules of M_1 are submodules of M_2 via f.

——— $N_3 \subseteq M_3$ come from $g^{-1}(N_3) \subseteq M_2$ submodules

$g^{-1}(N_3) = \langle m_1, \dots m_\ell \rangle$, then $N_3 = \langle g(m_1), \dots, g(m_\ell) \rangle$ (g is surjective!)

$\Leftarrow$) Pick N a submodule of M_2 then $g(N) \subseteq M_3$ is a subm so $g(N) = \langle m_1, \dots, m_s \rangle$

Pick $n_1, \dots n_s \in N$ with $g(n_1) = m_1$, $\dots$, $g(n_s) = m_s$

Next Take $f^{-1}(N \cap f(M_1)) = N_1 \subseteq M_1$ submodule, so fg, ie $N_1 = \langle q_1, \dots q_\ell \rangle$

Then $n'_1 = f(q_1)$, $\dots$, $n'_\ell = f(q_\ell)$. & $N \cap f(M_1) = \langle n'_1, \dots, n'_\ell \rangle$

Claim: $N = \langle n_1, \dots, n_s, n'_1, \dots, n'_\ell \rangle$

Pf/ Pick $n \in N$, so $g(n) \in g(N) = \langle m_1, \dots, m_s \rangle$ That is

$g(n) = a_1 m_1 + \dots + a_s m_s = a_1 g(n_1) + \dots + a_s g(n_s) = g(a_1 n_1 + \dots + a_s n_s)$ $\quad a_1, \dots a_s \in R$

$\Rightarrow m = n - a_1 n_1 - \dots - a_s n_s \in \operatorname{Ker} g = \operatorname{Im} f$, $\Rightarrow m \in N \cap f(M_1) = \langle n'_1, \dots n'_\ell \rangle$.

$\Rightarrow n \in \langle n_1, \dots, n_s, n'_1, \dots, n'_\ell \rangle$. $\square$

Obs: (1) • Submodules of a Noetherian module are Noetherian.

• Quotients of a Noetherian module are Noetherian

(2) A finite direct sum of Noetherian modules is Noetherian

(Hint: Induct on the number of summands & use Corollary 2)

(3) M: Noetherian R-module $\Rightarrow S^{-1}M$ is a Noetherian $S^{-1}R$-mod

S mult closed set of R

Proposition: Let R be a Noetherian ring & M an R-module. Then M is Noetherian if & only if M is finitely generated, ie $\exists\, x_1, \ldots, x_\ell \in M$ st every x in M can be written (no necessarily uniquely) as $x = a_1 x_1 + \cdots + a_\ell x_\ell$ for $a_1, \ldots, a_\ell \in R$.

Pf/ $(\Rightarrow)$ M is a submodule of M, so it's f.g. by Thm 2

($\Leftarrow$) Assume $M = \langle x_1, \dots x_\ell \rangle$ is f.g. R-module. Want to show: M is Noeth

Write $R \xrightarrow{f_i} M$ morphism of R-modules. By the
$a_i \longmapsto a_i x_i$

universal property of $\underbrace{R \oplus \dots \oplus R}_{\ell \text{ copies}}$ we have a unique

$$f : \bigoplus_{i=1}^{\ell} R \longrightarrow M \quad R\text{-linear map}$$
$$(a_1, \dots, a_\ell) \longmapsto \sum_{i=1}^{\ell} a_i x_i$$

Furthermore, we have a ses of R-modules:

$$0 \longrightarrow \ker f \longrightarrow \bigoplus_{i=1}^{\ell} R \xrightarrow{f} M \longrightarrow 0$$

But R is Noetherian, so $\bigoplus_{i=1}^{\ell} R$ is also a Noetherian R-module. Again by Corollary 2: $\operatorname{Im} f = M$ is Noetherian $\square$.

Examples

① $R = \mathbb{K}$ a field, $M = \mathbb{K}$-vector space

$$M \text{ Noetherian} \iff \dim_{\mathbb{K}} M < \infty$$

② $R = \mathbb{K}[x]$ (Noetherian ring, since it's a PID)

But $M = \mathbb{K}[x,y]$ is not a Noetherian R-module.

It will be a Noetherian ring! $(1, y, y^2, \dots)$ not fg R-submod.

③ An example of non-Noetherian ring:

$R = \{$continuous $\mathbb{C}$-valued functions on $\mathbb{R}\}$

$F_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$ $n \geq 1$ (nested chain of intervals with $|F_n| \searrow 0$)

$\mathfrak{a}_n = \{f \in R \mid f|_{F_n} \equiv 0\}$ is an ideal in R.

$\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \mathfrak{a}_3 \subsetneq \cdots\cdots$ is a strictly increasing chain of ideals.

So R is Not noetherian.

Hilbert Basis Theorem

Theorem 3: If R is commutative and Noetherian, so is $R[X]$.

Proof: We will show that every ideal of $R[X]$ is finitely gen.
Let $\mathfrak{b} \subset R[X]$ be an ideal. For every $f(x) \in R[X]$ we let $LT(f) \in R$ be the leading coefficient of f

- $\underset{\neq 0}{f} = a_0 + a_1 X + \dots + a_n x^n \rightsquigarrow LT(f) := a_n \in R$ with $a_n \neq 0$
- We define $LT(0) = 0$.

Claim: $\mathfrak{a} = \{LT(f) : f \in \mathfrak{b}\} \subset R$ is an ideal.

Pf/ (1) $0 \in \mathfrak{a}$ since $LT(0) = 0$ & $0 \in \mathfrak{b}$

(2) $a\,LT(f) \in \mathfrak{a}$ $\forall a \in R$ & $f \in \mathfrak{b}$. If $aLT(f) = 0$ ✓

Otherwise $aLT(f) = LT(af)$ ✓

$\rightsquigarrow$ $-LT(f) \in \mathfrak{a}$ if $LT(f) \in \mathfrak{a}$.

(3) $LT(f) + LT(g) \in \mathfrak{a}$ if $LT(f)$ & $LT(g)$ do

- If $LT(f) = -LT(g)$ we know $0 \in$
- If $LT(f) + LT(g) \neq 0$, assume $\underset{k}{\deg(f)} \leq \underset{l}{\deg g}$

then $x^{l-k} f \in \mathfrak{b}$ & $g \in \mathfrak{b}$, so $x^{l-k} f + g \in \mathfrak{b}$

& $LT(f) + LT(g) = LT(x^{l-k} f + g) \in \mathfrak{a}$

no cancellation occurs

$\Longrightarrow$ R Noetherian, gives $\mathfrak{a} = (a_1, \ldots, a_\ell)$, with $a_1, \ldots a_\ell \neq 0$.

For each $j = 1, \ldots, \ell$ pick $f_j \in \mathfrak{b}$ with $\boxed{a_j = LT(f_j).}$

Let $r = \max_{1 \leq j \leq \ell} \deg(f_j) \geq 0$. Need control over $f \in M$ with $\deg f < r$.

Set $M := $ submodule of $R[x]$ gen by $\{1, x, \ldots, x^{r-1}\} = \{f : \deg f < r\}$

M is fg & R Noeth $\Longrightarrow$ M is Noetherian R-module.

- $\mathfrak{b} \cap M \subset M$ submodule so f.g. $\mathfrak{b} \cap M = (b_1, \dots, b_k)$
- $\mathfrak{a} = \{LT(f) \;\; f \in \mathfrak{b}\} = (a_1, \dots, a_\ell)$ $\qquad LT(f_i) = a_i$

<u>Claim</u>: $\mathfrak{b} = (b_1 \dots b_k, f_1 \dots f_k)$ $\qquad \deg(b_i) \le r = \max\{\underbrace{\deg(f_j)}_{= d_j}\}$

Pf/ . If $f \in \mathfrak{b}$ $\deg f < r$, then $f \in \mathfrak{b} \cap M$ ✓

- Otherwise, induct on $\deg f \ge r$.

Let $a = LT(f)$ with $\deg(f) = d \ge \deg(f_j) =: d_j \;\forall j$

$a \in \mathfrak{a} \Rightarrow a = r_1 a_1 + \dots + r_\ell a_\ell$ for suitable $r_1, \dots, r_\ell$

Thus, $\boxed{g = f - \sum_{j=1}^{\ell} r_j x^{d-d_j} f_j \in \mathfrak{b}}$ & $\deg g < \deg f$.

- If $\deg f = r$, then $g \in \mathfrak{b} \cap M$ and we are done because

$$g = f - \sum r_j x^{d-d_j} f_j = c_1 b_1 + \dots + c_k b_k.$$

- If $\deg f > r$, then $\deg g < \deg f$ & $g \in \mathfrak{b}$. By IH, $g \in \langle b_1, \dots, b_k, f_1, \dots, f_\ell \rangle$, so the same holds for f. $\square$