Lecture 22: Artinian Rings I

Hilbert Basis Thm: If R is comm. and Noetherian, so is $R[X]$

Proof: We will show that every ideal of $R[X]$ is finitely gen.
Let $\mathfrak{b} \subset R[X]$ be an ideal. For every $f(x) \in R[X]$ we let $LT(f) \in R$ be the leading coefficient of f

- $\underset{\neq 0}{f} = a_0 + a_1 X + \cdots + a_n X^n \rightsquigarrow LT(f) := a_n \in R$ with $a_n \neq 0$
- We define $LT(0) = 0$.

Claim: $\mathfrak{a} = \{LT(f) : f \in \mathfrak{b}\} \subset R$ is an ideal.

Pf/ (1) $0 \in \mathfrak{a}$ since $LT(0) = 0$ & $0 \in \mathfrak{b}$

(2) $a\,LT(f) \in \mathfrak{a}$ $\forall a \in R$ & $f \in \mathfrak{b}$. If $aLT(f) = 0$ ✓
Otherwise $aLT(f) = LT(af)$ ✓
$\rightsquigarrow -LT(f) \in \mathfrak{a}$ if $LT(f) \in \mathfrak{a}$.

(3) $LT(f) + LT(g) \in \mathfrak{a}$ if $LT(f)$ & $LT(g)$ do

- If $LT(f) = -LT(g)$ we know $0 \in$
- If $LT(f) + LT(g) \neq 0$, assume $\underset{\overset{\parallel}{k}}{\deg(f)} \leq \underset{\overset{\parallel}{l}}{\deg g}$

then $x^{l-k} f \in \mathfrak{b}$ & $g \in \mathfrak{b}$, so $x^{l-k} f + g \in \mathfrak{b}$

& $LT(f) + LT(g) \underset{\uparrow}{=} LT(x^{l-k} f + g) \in \mathfrak{a}$

no cancellation occurs

Artinian Rings

Definition: Let R be a commutative ring. We say that R is Artinian (after Emil Artin) if every descending chain of ideals

$$\mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \dots$$

stabilizes, ie $\exists\ l \geq 0$ with $\mathfrak{a}_l = \mathfrak{a}_{l+1} = \dots$

Lemma 1: Let $\mathcal{S}$ be non-empty set of ideals in an Artinian ring. Then $\mathcal{S}$ has minimal elements (with respect to inclusion)

Example: 1. R field is Artinian

2. $R = \mathbb{K}[x]/(x^n)$ is Artinian

Lemma 2: Artinian property is preserved under quotients by ideals

Proposition 1: Let R be an Artinian commutative ring. Then:

(i) Every prime ideal in R is maximal.

(ii) There are only finitely many maximal ideals in R.

Proof of (i) To show Every prime ideal in R is maximal.

Proof of (ii)

$\boxed{\text{To show: } R \text{ only has finitely many maximal ideals}}$

Corollary: Let $\mathcal{N}$ be the ideal of nilpotent elements (ie the nilradical of R). If R is Artinian, then

$$\mathcal{N} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell$$

where $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell\}$ is the list of maximal ideals of R.

Obs Problem 9 in HW 7 says $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell = \mathcal{J}$, where

$$\mathcal{J} = \{x \in R : 1 - xy \text{ is a unit } \forall y \in R\}$$

is the Jacobson radical of R

Proposition 2: The ideal $\mathcal{N} \subset R$ (Artinian) is nilpotent, ie $\exists\ n \geq 0$ st $\mathcal{N}^n = (0)$