Lecture 22: Artinian Rings I

Hilbert Basis Thm: If R is comm. and Noetherian, so is $R[X]$

Proof: We will show that every ideal of $R[X]$ is finitely gen.
Let $\mathfrak{b} \subset R[X]$ be an ideal. For every $f(x) \in R[X]$ we let $LT(f) \in R$ be the leading coefficient of f

- $\underset{\neq 0}{f} = a_0 + a_1 X + \dots + a_n X^n \rightsquigarrow LT(f) := a_n \in R$ with $a_n \neq 0$
- We define $LT(0) = 0$.

Claim: $\mathfrak{a} = \{LT(f) : f \in \mathfrak{b}\} \subset R$ is an ideal.

Pf/ (1) $0 \in \mathfrak{a}$ since $LT(0) = 0$ & $0 \in \mathfrak{b}$

(2) $a\,LT(f) \in \mathfrak{a}$ $\forall a \in R$ & $f \in \mathfrak{b}$. If $a\,LT(f) = 0$ ✓

Otherwise $a\,LT(f) = LT(af)$ ✓

$\rightsquigarrow -LT(f) \in \mathfrak{a}$ if $LT(f) \in \mathfrak{a}$.

(3) $LT(f) + LT(g) \in \mathfrak{a}$ if $LT(f)$ & $LT(g)$ do

- If $LT(f) = -LT(g)$ we know $0 \in$
- If $LT(f) + LT(g) \neq 0$, assume $\underset{\overset{\shortparallel}{k}}{\deg(f)} \leq \underset{\overset{\shortparallel}{l}}{\deg g}$

then $x^{l-k} f \in \mathfrak{b}$ & $g \in \mathfrak{b}$, so $x^{l-k}f + g \in \mathfrak{b}$

& $LT(f) + LT(g) \underset{\uparrow}{=} LT(x^{l-k} f + g) \in \mathfrak{a}$

no cancellation occurs

$\Longrightarrow$ R Noetherian, gives $\mathfrak{a} = (a_1, \ldots, a_l)$, with $a_1, \ldots a_l \neq 0$.

For each $j = 1, \ldots, l$ pick $f_j \in \mathfrak{b}$ with $\boxed{a_j = LT(f_j).}$

Let $r = \max\limits_{1 \leq j \leq l} \deg(f_j) \geq 0$. Need control over $f \in M$ with $\deg f < r$.

Set $M := $ submodule of $R[x]$ gen by $\{1, x, \ldots, x^{r-1}\} = \{f : \deg f < r\}$

M is fg & R Noeth $\Longrightarrow$ M is Noetherian R-module.

• $\mathfrak{b} \cap M \subset M$ submodule so f.g. $\mathfrak{b} \cap M = (b_1, \dots, b_k)$

• $\mathfrak{a} = \{LT(f) \;\; f \in \mathfrak{b}\} = (a_1, \dots, a_\ell)$ $\qquad LT(f_i) = a_i$

<u>Claim</u>: $\mathfrak{b} = (b_1 \dots b_k, f_1 \dots f_k)$ $\qquad \deg(b_i) \leq r = \max\{\underbrace{\deg(f_j)}_{= d_j}\}$

Pf/ • If $f \in \mathfrak{b}$ $\deg f < r$, then $f \in \mathfrak{b} \cap M$ ✓

• Otherwise, induct on $\deg f \geq r$.

Let $a = LT(f)$ with $\deg(f) = d \geq \deg(f_j) =: d_j \; \forall j$

$a \in \mathfrak{a} \implies a = r_1 a_1 + \dots + r_\ell a_\ell$ for suitable $r_1 .. r_\ell$

Thus, $\boxed{g = f - \sum_{j=1}^{\ell} r_j x^{d-d_j} f_j \in \mathfrak{b}}$ & $\deg g < \deg f$.

• If $\deg f = r$, then $g \in \mathfrak{b} \cap M$ and we are done because

$$g = f - \sum r_j x^{d-d_j} f_j = c_1 b_1 + \dots + c_k b_k.$$

• If $\deg f > r$, then $\deg g < \deg f$ & $g \in \mathfrak{b}$. By IH,

$g \in \langle b_1, \dots, b_k, f_1, \dots, f_\ell \rangle$, so the same holds for f. $\square$

Artinian Rings

Definition: Let R be a commutative ring. We say that R is **Artinian** (after Emil Artin) if every descending chain of ideals

$$\mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$$

stabilizes, ie $\exists\ l \geq 0$ with $\mathfrak{a}_l = \mathfrak{a}_{l+1} = \cdots$ (Descending Chain Condition)

Geometrically Artinian rings correspond to finite collections of fat points (ie, points with multiplicities)

Lemma 1: Let $\mathcal{S}$ be non-empty set of ideals in an Artinian ring. Then $\mathcal{S}$ has minimal elements (with respect to inclusion)

Proof: (Same idea as with Noetherian rings)

Let $\mathfrak{a}_0 \in \mathcal{S}$. If $\mathfrak{a}_0$ is minimal, we are done. Otherwise, we find $\mathfrak{a}_1 \in \mathcal{S}$ with $\mathfrak{a}_0 \supsetneq \mathfrak{a}_1$. As R is Artinian, this process must stop and we will arrive at a minimal element of $\mathcal{S}$. $\square$

Example: ① R field is Artinian

② $R = \mathbb{K}[x]/(x^n)$ is Artinian (1 pt of multiplicity n)

(Ideals in R are also a $\mathbb{K}$-vector subspace since $\dim_{\mathbb{K}} R = n$, DCC holds)

Lemma 2: Artinian property is preserved under quotients by ideals

Pf/ Let $\mathfrak{a} \subseteq R$ be an ideal and R be Artinian

Then $\tilde{R} = R/\mathfrak{a}$ is also Artinian since ideals in $\tilde{R}$ correspond to ideals in R containing $\mathfrak{a}$.

So the DCC for R yields the DCC for $\tilde{R}$. $\square$

Proposition 1: Let R be an Artinian commutative ring. Then:

(i) Every prime ideal in R is maximal.

(ii) There are only finitely many maximal ideals in R.

Proof of (i) $\boxed{\text{To show: Every prime ideal in } R \text{ is maximal.}}$

- Let $\mathfrak{p} \subsetneq R$ be a prime ideal. Then $R/\mathfrak{p}$ is an Artinian integral domain. (by Lemma 2)

- Pick $x \in R/\mathfrak{p} \setminus \{0\}$ & consider the descending chain of ideals in $R/\mathfrak{p}$

$$(x) \supseteq (x^2) \supseteq (x^3)$$

DCC $\Rightarrow$ It stabilizes so $\exists\, k \geq 1$ with $(x^k) = (x^{k+1})$ ie

$$x^k = y x^{k+1} \text{ for } y \in R/\mathfrak{p}. \Rightarrow x^k(1 - xy) = 0$$

As $R/\mathfrak{p}$ is a domain and $x \neq 0 \Longrightarrow 1 = xy$, so x is a unit.

We conclude $(R/\mathfrak{p})^\times = R/\mathfrak{p} \setminus \{0\}$, so $R/\mathfrak{p}$ is a field.

This means $\mathfrak{p}$ is a maximal ideal of R. $\square$

<u>Proof of (ii)</u> $\boxed{\text{To show: } R \text{ only has finitely many maximal ideals}}$

Let $\boxed{\mathcal{J}}$ = {ideals that are finite intersections of max'l ideals of R}

- $\mathcal{J} \neq \emptyset$ since maximal ideals $\mathfrak{m}$ exist & $\mathfrak{m} \in \mathcal{J}$.
- By the Artinian condition & Lemma 1, $\mathcal{J}$ has a minimal element $\mathfrak{a} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell$

<u>Claim</u>: {Maximal ideals in R} $= \{\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell\}$

Pf/ Pick $\mathfrak{m} \subsetneq R$ maximal ideal, then $\mathfrak{m} \cap \mathfrak{a} \in \mathcal{J}$

& $\mathfrak{m} \cap \mathfrak{a} \subseteq \mathfrak{a}$ $\xRightarrow[\mathfrak{a} \text{ minimal}]{}$ $\mathfrak{a} = \mathfrak{m} \cap \mathfrak{a}$

so $\mathfrak{m} \cap \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell \subseteq \underset{\text{prime}}{\mathfrak{m}}$

By Prime Avoidance ($\underset{\text{Thm 2, page 7}}{\text{Lecture 17}}$) $\exists j$ st $\mathfrak{m}_j \subseteq \mathfrak{m}$

Since $\mathfrak{m}_j$ & $\mathfrak{m}$ are both maximal, we have $\mathfrak{m}_j = \mathfrak{m}$. $\square$

Corollary: Let $\mathcal{N}$ be the ideal of nilpotent elements (ie the nilradical of R). If R is Artinian, then

$$\mathcal{N} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell$$

where $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell\}$ is the list of maximal ideals of R.

Pf/ $\mathcal{N} \underset{\text{by Problem 4 HW7.}}{=} \bigcap\limits_{\substack{\mathfrak{p} \subseteq R \\ \text{prime}}} \mathfrak{p} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell$ because all prime ideals are max'l in the Artinian case

$\square$

Obs Problem 4 in HW7 says $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell = \mathcal{J}$, where

$$\mathcal{J} = \{x \in R : 1 - xy \text{ is a unit } \forall y \in R\}$$

is the Jacobson radical of R

Proposition 2: The ideal $\mathcal{N} \subset R$ (Artinian) is nilpotent, ie $\exists\, n \geq 0$ st $\mathcal{N}^n = (0)$

Proof: We consider the chain of ideals of R: $\mathcal{N} \supseteq \mathcal{N}^2 \supseteq \mathcal{N}^3 \supseteq \cdots$

Since R is Artinian, $\exists\, n \in \mathbb{N}$ with $\mathcal{N}^n = \mathcal{N}^{n+1} = \cdots =: \boxed{\mathfrak{a}}$

If $\mathfrak{a} = (0)$ we are done. So assume $\mathfrak{a} \neq (0)$. Consider.

$$\boxed{\mathcal{S} = \text{set of all ideals } \mathfrak{b} \subseteq R \text{ st } \mathfrak{a}\mathfrak{b} \neq (0)}$$

- $\mathcal{S} \neq \emptyset$ since $\mathfrak{a}^2 = \mathfrak{a} \neq (0)$ so $\mathfrak{a} \in \mathcal{S}$
- Pick $I \in \mathcal{S}$ minimal element. Since $I\mathfrak{a} \neq (0)$ $\exists\, x \in I$ with $x\mathfrak{a} \neq (0)$. So $(x) \in \mathcal{S}$ $\underset{\substack{I \text{ min} \\ x \in I}}{\Longrightarrow}$ $(x) = I$

But $(x\mathfrak{a})\mathfrak{a} = x\mathfrak{a}^2 = x\mathfrak{a} \neq \emptyset$ so $x\mathfrak{a} \in \mathcal{S}$ $\underset{x\mathfrak{a} \subseteq (x) \text{ min}}{\Longrightarrow}$ $(x) = x\mathfrak{a}$

This means $\exists\, y \in \mathfrak{a}$ st $x = xy$, ie $x = xy = xy^2 = \cdots$

Since $y \in \mathfrak{a} \subset \mathcal{N}$, then y is nilpotent, ie $\exists\, m \geq 1$ with $y^m = 0$

Conclusion: $x = xy^m = 0$, contradicting $x\mathfrak{a} = I\mathfrak{a} \neq (0)$ $\square$