Lecture 23: Artinian rings II

Last time: We defined Artinian rings R as rings with DCC

Proposition 1: Let R be an Artinian commutative ring. Then:

(i) Every prime ideal in R is maximal.

(ii) There are only finitely many maximal ideals in R

Property: R Artinian & integral domain $\Rightarrow$ R is a field.

Corollary: Nilradical = Jacobson radical for Artinian rings

Proposition 2: The nilradical ideal $\mathcal{N} \subset R$ (Artinian) is nilpotent, ie $\exists\, n \geq 0$ such that $\mathcal{N}^n = (0)$.

Examples: ① $\mathbb{K}$ $(\mathcal{N} = (0))$; ② $\mathbb{K}[x]/(x^n)$. $(\mathcal{N} = (x))$

TODAY: Structure Thm ; local Artinian rings & Hensel's lemma.

Geometric meaning of Artinian Rings

Lemma 3: Fix A a commutative ring and $\mathfrak{a}, \mathfrak{b} \subset A$ be coprime ideals. Then $\mathfrak{a}^i$ & $\mathfrak{b}^i$ are coprime $\forall i \geq 1$. Moreover $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \cdot \mathfrak{b}$

Proposition: $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ coprime, then $\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n = \mathfrak{a}_1 \cdots \mathfrak{a}_n$

We let $\boxed{\mathfrak{m}_1, \dots, \mathfrak{m}_\ell}$ be the maximal ideals (= prime ideals) of R

- Pick $n \in \mathbb{N}$ with $\mathcal{N}^n = (0)$ (ok by Prop 2)
- $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ are coprime $\Longrightarrow$ $\mathfrak{m}_1^n, \dots, \mathfrak{m}_\ell^n$ are coprime

Theorem: $R \simeq R/\mathfrak{m}_1^n \times R/\mathfrak{m}_2^n \times \dots \times R/\mathfrak{m}_\ell^n$

Here, $R/\mathfrak{m}_j^n$ is a local ring with unique maximal ideal

$\overline{\mathfrak{m}_j} = \pi_j(\mathfrak{m}_j)$ where $\pi_j : R \twoheadrightarrow R/\mathfrak{m}_j^n$ is the natural projection.

Next: We study each $R/\mathfrak{m}_j^n$ = local Artinian rings

(quotient of Artinian ring, so Artinian)

Local Artinian rings

Proposition: If R is Artinian and local, then R is Noetherian

Corollary: R Artinian $\Rightarrow$ R Noetherian

Pf/. $R \xrightarrow{\sim} R/m_1^n \times \cdots \times R/m_l^n$

- Direct finite sum of Noetherians is Noetherian.

Hensel's Lemma: Let (R, m) be an Artinian local ring.

Let $f(x) \in R[X]$ and assume we have $g(x), h(x)$ in $R[X]$ s.t.

- $f(x) - g(x)h(x) \in m[x] = mR[x]$
- $g(x)$ & $h(x)$ are coprime modulo m, i.e. $\exists\, a(x), b(x) \in R[x]$ s.t. $ag + bh = 1$ in $(R/m)[x]$

Then, $\exists\, \tilde{g}, \tilde{h} \in R[X]$ st $f(x) = g(x)h(x)$ and
$g(x) \equiv \tilde{g}(x)$ & $h(x) \equiv \tilde{h}(x)$ modulo $m[x]$.

Proof: We will build approximations of g & h working modulo m^{ℓ}.