Lecture 23: Artinian rings II

Last time: We defined Artinian rings R as rings with DCC

Proposition 1: Let R be an Artinian commutative ring. Then:

(i) Every prime ideal in R is maximal.

(ii) There are only finitely many maximal ideals in R

Property: R Artinian & integral domain $\Rightarrow$ R is a field.

Corollary: Nilradical = Jacobson radical for Artinian rings

Proposition 2: The nilradical ideal $\mathcal{N} \subset R$ (Artinian) is nilpotent, ie $\exists\, n \geq 0$ such that $\mathcal{N}^n = (0)$.

Examples: ① $\mathbb{K}$ $(\mathcal{N} = (0))$; ② $\mathbb{K}[x]/(x^n)$. $(\mathcal{N} = (x))$

TODAY: Structure Thm ; local Artinian rings & Hensel's lemma.

$\boxed{\text{Geometric meaning of Artinian Rings}}$

Lemma 3: Fix A a commutative ring and $\mathfrak{a}, \mathfrak{b} \subset A$ be coprime ideals. Then $\mathfrak{a}^i$ & $\mathfrak{b}^i$ are coprime $\forall i \geq 1$. Moreover $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \cdot \mathfrak{b}$

Proof: Since $\mathfrak{a} + \mathfrak{b} = 1$ $\exists\, a \in \mathfrak{a}$ & $b \in \mathfrak{b}$ with $1 = a + b$

$$1 = (a+b)^{2i} = \sum_{j=0}^{2i} \binom{2i}{j} a^{2i-j} b^j$$

$$= \underbrace{\left(\sum_{j=0}^{i} \binom{2i}{j} b^j a^{i-j}\right) a^i}_{\in \mathfrak{a}^i} + \underbrace{\left(\sum_{j=1}^{i} \binom{2i}{j+i} a^{i-j} b^j\right) b^i}_{\in \mathfrak{b}^i}$$

- $\mathfrak{a} \cdot \mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$ is always true
- Pick $x \in \mathfrak{a} \cap \mathfrak{b}$ $\quad 1 = a + b$ so $x = \underbrace{a\overset{\in \mathfrak{b}}{x}}_{\in \mathfrak{a}\mathfrak{b}} + \underbrace{\overset{\in \mathfrak{a}}{x}b}_{\in \mathfrak{a}\mathfrak{b}} \in \mathfrak{a}\mathfrak{b}$ $\square$

Proposition: $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ coprime, then $\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n = \mathfrak{a}_1 \cdots \mathfrak{a}_n$

Proof: Same as above.

We let $\boxed{\mathfrak{m}_1, \dots, \mathfrak{m}_\ell}$ be the maximal ideals (= prime ideals) of R

- Pick $n \in \mathbb{N}$ with $\mathcal{N}^n = (0)$ (ok by Prop 2)
- $\mathfrak{m}_1, \dots \mathfrak{m}_\ell$ are coprime $\Rightarrow$ $\mathfrak{m}_1^n, \dots, \mathfrak{m}_\ell^n$ are coprime

Theorem: $R \cong R/\mathfrak{m}_1^n \times R/\mathfrak{m}_2^n \times \dots \times R/\mathfrak{m}_\ell^n$

Here, $R/\mathfrak{m}_j^n$ is a local ring with unique maximal ideal

$\overline{\mathfrak{m}}_j = \pi_j(\mathfrak{m}_j)$ where $\pi_j : R \twoheadrightarrow R/\mathfrak{m}_j^n$ is the natural projection.

Proof: CRT $\Rightarrow$ $\varphi : R \twoheadrightarrow R/\mathfrak{m}_1^n \times \dots \times R/\mathfrak{m}_\ell^n$

$r \longmapsto (\pi_1(r), \dots, \pi_\ell(r))$

$\operatorname{Ker}\varphi = \mathfrak{m}_1^n \cap \dots \cap \mathfrak{m}_\ell^n \underset{(L3)}{=} \mathfrak{m}_1^n \cdots \mathfrak{m}_\ell^n \underset{(*)}{\subseteq} \mathcal{N}^n = (0)$ $\Rightarrow$ φ injective

$\Rightarrow$ φ bij & ring hom, so iso.

(*) $\mathcal{N} = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n \overset{(L3)}{=} \mathfrak{m}_1 \cdots \mathfrak{m}_\ell$

It remains to show that $R/\mathfrak{m}_j^n$ is local $\forall\, 1 \le j \le l$.

Let $\mathfrak{q} \subsetneq R/\mathfrak{m}_j^n$ be a maximal ideal, so it's prime.

Then $\mathfrak{q}$ is the image of $\mathfrak{p} = \pi^{-1}(\mathfrak{q}) \subseteq R$ & $\mathfrak{p}$ is a prime ideal containing $\mathfrak{m}_j^n$.

Problem 12 HW7: $\mathfrak{m}^n \subseteq \mathfrak{p}$ $\mathfrak{m}$ maximal & $\mathfrak{p}$ prime $\Rightarrow \mathfrak{m} = \mathfrak{p}$.

We conclude $\mathfrak{m}_j = \mathfrak{p}$ & so $\mathfrak{q} = \pi(\mathfrak{m}_j)$ $\square$

Next: We study each $R/\mathfrak{m}_j^n$ = local Artinian rings
(quotient of Artinian ring, so Artinian)

Local Artinian rings

<u>Proposition</u>: If R is Artinian and local, then R is Noetherian

Pf/ Set $\mathfrak{m}$ = unique mxl ideal of R & write $k = R/\mathfrak{m}$ for the quotient.
field

For each $j \geq 0$: $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is a k-module ($=$ v.s.)

$$k = R/\mathfrak{m} \underset{\substack{\uparrow \\ 2^{nd}\text{ iso Thm}}}{=} \left(R/\mathfrak{m}^{j+1}\right)\Big/\left(\mathfrak{m}/\mathfrak{m}^{j+1}\right)$$

<u>Claim 1</u>: $\boxed{\dim_k \mathfrak{m}^j/\mathfrak{m}^{j+1} < \infty}$.

Pf/ $\{k\text{-subspaces of } \mathfrak{m}^j/\mathfrak{m}^{j+1}\} \xleftrightarrow{\text{1-to-1}} \{I \subset R/\mathfrak{m}^{j+1} \text{ ideal with } I \subseteq \mathfrak{m}^j/\mathfrak{m}^{j+1}\}$

<u>Why?</u> $R \overset{\pi}{\twoheadrightarrow} R/\mathfrak{m}^{j+1}$ $\quad I \subset \mathfrak{m}^j/\mathfrak{m}^{j+1}$ is R-mod via $r x = \bar{r}\cdot x$, so ideal

Then: $\mathfrak{m}^{j+1} \subseteq \underset{=: J}{\pi^{-1}(I)} \subset \pi^{-1}\left(\mathfrak{m}^j/\mathfrak{m}^{j+1}\right) \underset{(*)}{=} \mathfrak{m}^j + \mathfrak{m}^{j+1} = \mathfrak{m}^j \Rightarrow \mathfrak{m}/\mathfrak{m}^{j+1} \cdot I = (0)$
$\hookrightarrow \mathfrak{m}^{j+1} \subseteq \mathfrak{m}^j$

Converse is clear: $\mathfrak{m}/\mathfrak{m}^{j+1} \cdot I = (0)$ so I is k-module.

If $\dim_k \mathfrak{m}^j/\mathfrak{m}^j$ can find an infinite str desc sequences of k-subspaces (take countable set of l.i vectors $\{v_n\}_{n\in\mathbb{N}}$ & set $W_i = \mathrm{Sp}(v_m : m \geq i)$)

Then we get an infinite desc chain of ideals in $R/\mathfrak{m}^{j+1}$, contr! (it's Artinian)

<u>To show</u>: $\mathfrak{a} \subsetneq R$ ideal is fg.

Consider $\boxed{\mathfrak{a}_j} = \mathfrak{a} \cap \mathfrak{m}^j$ for all $j \geq 1$ so $\mathfrak{a}_j \cap \mathfrak{m}^{j+1} = \mathfrak{a}_{j+1}$

Use
$$\begin{tikzcd} \mathfrak{a}_j \arrow[r, hook] \arrow[rr, bend right, "f"'] & \mathfrak{m}^j \arrow[r, "\pi"] & \mathfrak{m}^j/\mathfrak{m}^{j+1} \end{tikzcd}$$
$\Rightarrow \ker(f) = \mathfrak{a}_j \cap \mathfrak{m}^{j+1} = \mathfrak{a}_{j+1}$

So $\mathfrak{a}_j/\mathfrak{a}_{j+1}$ is a k-subspace of $\mathfrak{m}^j/\mathfrak{m}^{j+1}$, so finite dimensional by Claim

Let $\{\bar{a}_1^{(j)}, \ldots, \bar{a}_{\alpha_j}^{(j)}\}$ be a k-basis for $\mathfrak{a}_j/\mathfrak{a}_{j+1}$, with $a_i^{(j)} \in \mathfrak{a}_j$ $\forall i$.

Set $\boxed{\tilde{\mathfrak{a}}}$ = ideal in R generated by $\bigcup_{j \geq 1} \{a_1^{(j)}, \ldots, a_{\alpha_j}^{(j)}\}$ (fg since $\mathfrak{a}_n = (0)$ b/c $\mathfrak{m}^n = (0)$)

We show $\tilde{\mathfrak{a}} = \mathfrak{a}$ (so $\mathfrak{a}$ is fg), via:

<u>Claim 2</u>: $\tilde{\mathfrak{a}}_j := \tilde{\mathfrak{a}} \cap \mathfrak{m}^j$ equals $\mathfrak{a}_j$ for $j = 1, \ldots, n$ ($\mathfrak{a}_1 = \mathfrak{a}$ & $\tilde{\mathfrak{a}}_1 = \tilde{\mathfrak{a}}$)

Pf/ Reverse induction on j. $\tilde{\mathfrak{a}}_n = (0) = \mathfrak{a}_n$. is the base case

<u>Inductive Step</u>: $\mathfrak{a}_j \cap \tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}_j$ & $\tilde{\mathfrak{a}}_j/\tilde{\mathfrak{a}}_{j+1} = \mathfrak{a}_j/\mathfrak{a}_{j+1}$ (basis in $\tilde{\mathfrak{a}}_j/\mathfrak{a}_{j+1}$)

so $\mathfrak{a}_j = \tilde{\mathfrak{a}}_j$.

Corollary: R Artinian $\Rightarrow$ R Noetherian

Pf/. $R \xrightarrow{\sim} R/m_1^n \times \cdots \times R/m_\ell^n$

Artinian + Local, so Noetherian

- Direct finite sum of Noetherians is Noetherian.

Hensel's Lemma: Let (R, m) be an Artinian local ring.

Let $f(x) \in R[X]$ and assume we have $g(x), h(x)$ in $R[X]$ s.t.

- $f(x) - g(x)h(x) \in m[x] = mR[x]$
- $g(x)$ & $h(x)$ are coprime modulo m, i.e $\exists\, a(x), b(x) \in R[x]$

s.t. $ag + bh = 1$ in $(R/m)[x]$

Then, $\exists\, \tilde{g}, \tilde{h} \in R[X]$ st $f(x) = \tilde{g}(x)\tilde{h}(x)$ and

$g(x) \equiv \tilde{g}(x)$ & $h(x) \equiv \tilde{h}(x)$ modulo $m[x]$.

Proof: We will build approximations of g & h working modulo $\mathfrak{m}^\ell$.

Claim $\exists\, g_\ell, h_\ell \in R[x]$ st
- $f - g_\ell h_\ell \in \mathfrak{m}^\ell[x]$,
- $g_\ell - g \in \mathfrak{m}[x]$ & $h_\ell - h \in \mathfrak{m}[x]$

Pf/ By induction on ℓ

Base case: $\ell = 1$ Take $g_1 = g$ & $h_1 = h$ (statement's hypothesis)

Inductive Step: Assume we've constructed $\{g_\ell, h_\ell\}$. Let

$$\boxed{c_\ell(x) = f(x) - g_\ell(x)\, h(x)} \in \mathfrak{m}^\ell[x].$$

Set
$$\begin{cases} g_{\ell+1} = g_\ell + c_\ell\, b & \rightsquigarrow\ g_\ell - g_{\ell+1} \in \mathfrak{m}^\ell[x] \subseteq \mathfrak{m}[x] \\ h_{\ell+1} = h_\ell + c_\ell\, a & \rightsquigarrow\ h_\ell - h_{\ell+1} \in \mathfrak{m}^\ell[x] \subseteq \mathfrak{m}[x] \end{cases}$$

$$\Longrightarrow f(x) - g_{\ell+1} h_{\ell+1} = \overbrace{f(x) - g_\ell h_\ell}^{=c_\ell} - c_\ell (a g_\ell + b h_\ell) + c_\ell^2 ab$$

$$= \underbrace{c_\ell\,(1 - \underbrace{(ag+bh)}_{\in 1+\mathfrak{m}[x]})}_{\in \mathfrak{m}^\ell \mathfrak{m}[x] = \mathfrak{m}^{\ell+1}[x]} + \underbrace{\underbrace{c_\ell}_{\in \mathfrak{m}^\ell[x]} a \underbrace{(g - g_\ell)}_{\in \mathfrak{m}[x]}}_{\in \mathfrak{m}^{\ell+1}[x]} + \underbrace{\underbrace{c_\ell}_{\in \mathfrak{m}^\ell[x]} b \underbrace{(h - h_\ell)}_{\in \mathfrak{m}[x]}}_{\in \mathfrak{m}^{\ell+1}[x]} + \underbrace{c_\ell^2 ab}_{\in \mathfrak{m}^{2\ell}(x) \subseteq \mathfrak{m}^{\ell+1}(x)}$$

$$\Longrightarrow f - g_{\ell+1} h_{\ell+1} \in \mathfrak{m}^{\ell+1}[x]$$

- $g_{\ell+1} - g = \underbrace{(g_{\ell+1} - g_\ell)}_{\in \mathfrak{m}^\ell[x]} + \underbrace{(g_\ell - g)}_{\in \mathfrak{m}[x] \text{ by IH}} \in \mathfrak{m}[x]$ ✓

- $h_{\ell+1} - h = \underbrace{(h_{\ell+1} - h_\ell)}_{\in \mathfrak{m}^\ell[x]} + (h_\ell - h) \in \mathfrak{m}[x]$ ✓ $\square$

Using the claim for $\ell = n$ with $\mathfrak{m}^n = (0)$ we get

- $f - g_n h_n \in \mathfrak{m}^0[x] \rightsquigarrow f = g_n h_n$.
- $g_n - g \in \mathfrak{m}[x]$ & $h_n - h \in \mathfrak{m}[x]$

Thus $\tilde{g} = g_n$ & $\tilde{h} = h_n$ satisfy the desired properties. $\square$