Lecture 24: Artinian Rings & Primary Decomposition

Last Time: $\boxed{\text{Artinian} \implies \text{dimension } 0 \text{ \& Noetherian}}$ (*)

Structure Theorem: finitely many max'l ideals $(\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell)$ and
(R Comm & Art)
$$R \simeq R/\mathfrak{m}_1^n \times \cdots \times R/\mathfrak{m}_\ell^n$$
(each $R/\mathfrak{m}_j^n$ is Artinian & local)

TODAY we will show the converse to (*). The proof is based on "primary decomp".

Definition: An ideal $\mathfrak{q} \subsetneq R$ is **primary** if for any $a, b \in R$ we have
" $ab \in \mathfrak{q}$ & $b \notin \mathfrak{q} \implies a^n \in \mathfrak{q}$ for some $n \geq 1$."

Recall: The **radical** of an ideal $\mathfrak{a}$ in R is
$$r(\mathfrak{a}) = \sqrt{\mathfrak{a}} = \{a \in R \mid a^n \in \mathfrak{a} \text{ for some } n \text{ in } \mathbb{N}\}$$

Lemma: $\mathfrak{q} \subseteq R$ primary $\implies r(\mathfrak{q})$ is prime

$\boxed{\text{Examples}}$

① $R = \mathbb{K}[x]$ $\quad (x^n)$ is primary.

② $R = \mathbb{K}[x,y,z]/(xy - z^2) \supset \mathfrak{p} = (\bar{x}, \bar{z})$

<u>Claim</u>: $\mathfrak{p}$ is a prime ideal but $\mathfrak{p}^2$ is not primary.

③ $R = \mathbb{K}[x,y]$ $\quad \mathfrak{q} = (x, y^2)$ is primary (*) but <u>NOT</u> a power of a prime ideal.

Proposition: If R is commutative & $r(\mathfrak{q})$ is maximal then $\mathfrak{q}$ is primary. (HW 9)

Summary of examples:

- $\mathfrak{q}$ primary $\not\Rightarrow$ $\mathfrak{q}$ = power of a prime ideal
- $\mathfrak{p}$ prime $\not\Rightarrow$ $\mathfrak{p}^n$ primary.
- $r(\mathfrak{q})$ is maximal $\Rightarrow$ $\mathfrak{q}$ primary

Obs: The difference between $\mathfrak{q}$ & $\mathfrak{p} = r(\mathfrak{q})$ is algebraic and highlights the difference between a fat point (point with multiplicity) vs the point as a set. (The "algebraic part" of Alg Geometry)

Irreducible ideals

Def: An ideal $\mathfrak{a} \subseteq R$ is <u>irreducible</u> if $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ with $\mathfrak{b}, \mathfrak{c} \subseteq R$ ideals, then $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{c}$.

- Terminology comes from topology: if $R = \mathbb{C}[x_1, \dots x_n]$, then $\mathfrak{a}$, $\mathfrak{b}$ & $\mathfrak{c}$ define closed sets in $\mathbb{C}^n$ (solutions to polynomials in each ideal): $V(\mathfrak{a}), V(\mathfrak{b})$ & $V(\mathfrak{c})$. Moreover: $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \implies V(\mathfrak{a}) = V(\mathfrak{b}) \cup V(\mathfrak{c})$ decomp

Lemma: Assume R is <u>Noetherian</u>. Then:

(i) Every ideal in R is a finite intersection of irreducible ideals.

(ii) Irreducible $\implies$ Primary

(ii) To show Irred $\Rightarrow$ Primary

Fix $\mathfrak{a} \subsetneq R$ irreducible ideal

More on Primary Ideals

Fix R Noetherian & commutative. Let $\mathfrak{a} \subsetneq R$ be an ideal.

Write $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_\ell$ (a primary decomposition of $\mathfrak{a}$)

irreducible ($\Rightarrow$ primary)

Let $\mathfrak{p}_i = r(\mathfrak{q}_i)$ be the corresponding prime ideals.

Lemma: If $\mathfrak{p} \subsetneq R$ is a prime ideal, minimal among the set of prime ideals containing $\mathfrak{a}$, then $\mathfrak{p} = \mathfrak{p}_i$ for some $i = 1, \dots, \ell$

Def: The **minimal primes** of R are the prime ideals of R, minimal with respect to inclusion.

Corollary: There are only finitely many minimal primes over any given ideal $\mathfrak{a}$ of a Noetherian ring R. ($\longleftrightarrow$ min primes in $R/\mathfrak{a}$)

Theorem: Let R be a commutative Noetherian ring R of dim 0 (that is, every prime ideal is maximal). Then, R is Artinian.