Lecture 24: Artinian Rings & Primary Decomposition

Last Time: $\boxed{\text{Artinian} \Longrightarrow \text{dimension } 0 \text{ \& Noetherian}}$ $(*)$

Structure Theorem: finitely many max'l ideals $(\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell)$ and
(R Comm & Art) $R \simeq R/\mathfrak{m}_1^n \times \cdots \times R/\mathfrak{m}_\ell^n$ (each $R/\mathfrak{m}_j^n$ is Artinian & local)

TODAY we will show the converse to $(*)$. The proof is based on "primary decomp"

Definition: An ideal $q \subsetneq R$ is **primary** if for any $a, b \in R$ we have
" $ab \in q$ & $b \notin q \Longrightarrow a^n \in q$ for some $n \geq 1$."

Obs: Equivalently, every zero divisor in R/q is nilpotent. (HW9)

Recall: The **radical** of an ideal $\mathfrak{a}$ in R is
$$r(\mathfrak{a}) = \sqrt{\mathfrak{a}} = \{a \in R \mid a^n \in \mathfrak{a} \text{ for some } n \text{ in } \mathbb{N}\}$$

Lemma: $q \subseteq R$ primary $\Longrightarrow r(q)$ is prime

Pf/ $a, b \in \mathfrak{p} = r(q) \Longrightarrow a^k b^k \in q$ for some $k > 0 \Rightarrow$ Either $b^k \in q$ (so $b \in \mathfrak{p}$) or $(a^k)^N \in q$ for some N, so $a \in \mathfrak{p}$. $\square$

Examples

① $R = \mathbb{K}[x]$ $\quad (x^n)$ is primary. $r((x^n)) = (x)$ prime

Pf/ $ab \in (x^n)$ then $x \mid a$ or $x \mid b$. This forces $a^n \in (x^n)$ or $b^n \in (x^n)$ because if $x^n \nmid b$, then $x \mid a$.

② $R = \mathbb{K}[x,y,z]/(xy - z^2) \supset \mathfrak{p} = (\bar{x}, \bar{z})$

<u>Claim</u>: $\mathfrak{p}$ is a prime ideal but $\mathfrak{p}^2$ is not primary.

- $R/\mathfrak{p} = \mathbb{K}[x,y,z]/(x, z, xy - z^2) = \dfrac{\mathbb{K}[x,y,z]}{(x,z)} = \mathbb{K}[y]$ integral domain
- $\mathfrak{p}^2 = (\bar{x}^2, \bar{z}^2, \bar{x}\bar{z})$

$\bar{z}^2 = \bar{y}\bar{x} \in \mathfrak{p}^2$ & $\bar{x} \notin \mathfrak{p}^2$ but $\bar{y} \notin \underbrace{r(\mathfrak{p}^2)}_{= (\bar{x}, \bar{z}) = \mathfrak{p}}$. (Exercise)

⚠ We do have $\bar{y} \notin \mathfrak{p}^2$ but $\bar{x}^2 \in \mathfrak{p}^2$, i.e., the definition of primary is not symmetric in f & g.

③ $R = \mathbb{K}[x,y]$ $\quad \mathfrak{q} = (x, y^2)$ is primary(*) but **NOT** a power of a prime ideal.

$\mathfrak{p} = r(\mathfrak{q}) = (x,y)$

$x \notin \mathfrak{p}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$

Obs $r(\mathfrak{q})$ is maxl.

(*) $f = a_0 + x f_{1(x,y)} + y f_{2}(y) \quad a_0 \in \mathbb{K}$
$\quad g = b_0 + x g_{1(x,y)} + y g_{2(y)} \quad b_0 \in \mathbb{K}$ $\quad$ & $fg \in \mathfrak{q}$

$g \notin \mathfrak{q}$ means $b_0 \neq 0$ or $(f_3(0) \neq 0$ and $b_0 = 0)$

• Case 1: $b_0 \neq 0 \Rightarrow fg = \boxed{a_0 b_0} + x(g f_1 + a_0 g_1 + g_1 y f_{2(y)}) + y(f_2 g + a_0 g_{2(y)})$

$=0$ so $a_0 = 0$

$\Rightarrow \underbrace{fg}_{\in \mathfrak{q}} = \underbrace{x(f_1 g + g_1 f_{2(y)} g_1) + y^2 f_2 g_{2(y)}}_{\in \mathfrak{q}} + b_0 y f_{2(y)} \Rightarrow b_0 y f_{2(y)} \in \mathfrak{q}$

So $b_0 y f_{2(y)} = x h_1(x,y) + y^2 h_2(x,y) \Rightarrow$ Set $x=0$: $b_0 y f_{2(y)} = y^2 h_2(0,y)$

$\Rightarrow y \mid f_2(y)$ so $f = 0 + x f_1(x,y) + y^2 \underbrace{\frac{f_2}{y}}_{\in R} \in \mathfrak{q}$.

• Case 2: $b_0 = 0$ & $y \nmid g_2(y)$

$\Rightarrow \underbrace{fg}_{\in \mathfrak{q}} = \underbrace{x(a_0 g_1 + x f_1 g_1 + y f_{2}(y) g_1 + y f_1 g_{2}(y)) + y^2 f_2 g_2}_{\in \mathfrak{q}} + y a_0 g_2(y)$

$\Rightarrow y a_0 g_2(y) \in \mathfrak{q}$ so $y a_0 g_{2(y)} = x h_1(x,y) + y^2 h_2(x,y) \xRightarrow{x=0} a_0 = 0 \; (y \nmid g_{2}(y))$

$\Rightarrow f = x f_1 + y f_{2(y)} \Rightarrow f^2 \in \mathfrak{q}$

Proposition: If R is commutative & $r(q)$ is maximal then q is primary. (HW 9)

Summary of examples:

- q primary $\not\Longrightarrow$ q = power of a prime ideal
- $\mathfrak{p}$ prime $\not\Longrightarrow$ $\mathfrak{p}^n$ primary.
- $r(q)$ is maximal $\Longrightarrow$ q primary

Obs: The difference between q & $\mathfrak{p} = r(q)$ is algebraic and highlights the difference between a fat point (point with multiplicity) vs the point as a set. (The "algebraic part" of Alg Geometry)

Irreducible ideals

Def: An ideal $\mathfrak{a} \subseteq R$ is <u>irreducible</u> if $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ with $\mathfrak{b}, \mathfrak{c} \subseteq R$ ideals, then $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{c}$.

- Terminology comes from topology: if $R = \mathbb{C}[x_1, \ldots x_n]$, then $\mathfrak{a}$, $\mathfrak{b}$ & $\mathfrak{c}$ define closed sets in $\mathbb{C}^n$ (solutions to polynomials in each ideal): $V(\mathfrak{a}), V(\mathfrak{b})$ & $V(\mathfrak{c})$. Moreover: $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \implies V(\mathfrak{a}) = V(\mathfrak{b}) \cup V(\mathfrak{c})$ decomp

Lemma: Assume R is <u>Noetherian</u>. Then:

(i) Every ideal in R is a finite intersection of irreducible ideals.

(ii) Irreducible $\Rightarrow$ Primary

Proof: (i) Consider $\Sigma = \{\mathfrak{a} \subseteq R \text{ ideal} : \mathfrak{a} \neq \text{finite intersection of irred. ideals}\}$

- If $\Sigma \neq \emptyset$ it must have a maximal element (R Noeth), say $\mathfrak{a}$, so NOT irred.

Write $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ with $\mathfrak{a} \subsetneq \mathfrak{b}$ & $\mathfrak{a} \subsetneq \mathfrak{c}$ ideals. Now $\mathfrak{b}, \mathfrak{c} \notin \Sigma$ by maximality of $\mathfrak{a}$, so $\mathfrak{b} = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_k$, $\mathfrak{c} = \mathfrak{c}_1 \cap \cdots \cap \mathfrak{c}_l$ irreds $\Rightarrow \mathfrak{a} = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{c}_l \in \Sigma$ Contra!

(ii) To show Irred $\Rightarrow$ Primary. Fix $\mathfrak{a} \subsetneq R$ irreducible ideal.

Working with $\tilde{R} = R/\mathfrak{a}$ (still Noetherian), we may assume (0) is an irreducible ideal (Useful Trick)

Let $xy \in (0)$ & $y \notin (0)$ ie $xy = 0$ & $y \neq 0$. We want to prove $x^n = 0$ for some $n > 0$.

Consider the chain of ideals: $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x^2) \subseteq \cdots$

$[\, \mathrm{Ann}(z) = \{ r \in R \mid rz = 0 \} \subset_{\text{ideal}} R \,]$.

$\xRightarrow{R \text{ Noeth}}$ $\exists\, n > 0$ st $\mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1}) = \cdots$

Claim: $(0) = (x^n) \cap (y)$.

Pf/$\supseteq$) $\left.\begin{array}{l} a \in (y) \Rightarrow ax = 0 \quad (\text{since } xy = 0) \\ a \in (x^n) \Rightarrow a = bx^n \end{array}\right\} \Rightarrow bx^{n+1} = 0.$

But $bx^{n+1} = 0 \Rightarrow b \in \mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$ so $bx^n = 0 \Rightarrow a = bx^n = 0$

Since (0) is irreducible and $(y) \neq (0)$, we conclude $(x^n) = (0)$, ie $x^n = 0$ as required $\square$

This lemma is referred to as "Primary Decomposition for Noetherian rings".

More on Primary Ideals

Fix R Noetherian & commutative. Let $\mathfrak{a} \subsetneq R$ be an ideal.

Write $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\ell$ (a primary decomposition of $\mathfrak{a}$)

irreducible ($\Rightarrow$ primary)

Let $\mathfrak{p}_i = r(\mathfrak{q}_i)$ be the corresponding prime ideals.

Lemma: If $\mathfrak{p} \subsetneq R$ is a prime ideal, minimal among the set of prime ideals containing $\mathfrak{a}$, then $\mathfrak{p} = \mathfrak{p}_i$ for some $i = 1, \ldots, \ell$

Pf/ By Theorem of Prime Avoidance (Lecture 17), we have $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\ell \subseteq \mathfrak{p} \Rightarrow \mathfrak{q}_i \subseteq \mathfrak{p}$ for some i.

prime

Hence $\mathfrak{p}_i = r(\mathfrak{q}_i) \subseteq r(\mathfrak{p}) = \mathfrak{p}$, but

$$\mathfrak{a} \subseteq \mathfrak{p}_i \subseteq \mathfrak{p} \quad \& \ \mathfrak{p} \text{ minimal} \Rightarrow \mathfrak{p}_i = \mathfrak{p}. \qquad \square$$

prime

Def: The <u>minimal primes</u> of R are the prime ideals of R, minimal with respect to inclusion

Corollary: There are only finitely many minimal primes over any given ideal $\mathfrak{a}$ of a Noetherian ring R. ($\longleftrightarrow$ min primes in $R/\mathfrak{a}$)

Theorem: Let R be a commutative Noetherian ring R of dim 0 (that is, every prime ideal is maximal). Then, R is Artinian.

Proof: Next Time