R Commutative

Lecture 25: Primary Decomposition II

Recall • $\mathfrak{q}$ primary ideal if $xy \in \mathfrak{q}$ & $y \notin \mathfrak{q} \Rightarrow x^n \in \mathfrak{q}$ for some n

• $\mathfrak{q}$ irred ideal if $\mathfrak{q} = \mathfrak{b} \cap \mathfrak{c}$ with $\mathfrak{b}, \mathfrak{c}$ ideals $\Rightarrow \mathfrak{q} = \mathfrak{b}$ or $\mathfrak{q} = \mathfrak{c}$.

Lemma: $\mathfrak{q}$ primary $\Longrightarrow$ $r(\mathfrak{q})$ prime

Obs: ① $\mathfrak{q}$ primary $\not\Longrightarrow$ $\mathfrak{q}$ is a power of a prime

② $\mathfrak{p}$ prime $\not\Longrightarrow$ $\mathfrak{p}^n$ is primary

③ $r(\mathfrak{q})$ maximal $\Longrightarrow$ $\mathfrak{q}$ is primary

Prop: R Noetherian : (i) Every ideal is a finite intersection of irred. ideals

(ii) Irred $\Longrightarrow$ Primary.

• **Consequence**: "Primary Decomposition" for Noetherian rings.

• $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ & $\mathfrak{p}$ minimal among prime ideals containing $\mathfrak{a}$

then $\mathfrak{p} = r(\mathfrak{q}_i)$ for some i

Consequence: $\mathfrak{a}$ has finitely many minimal associate primes. $\boxed{= \mathrm{Min}(\mathfrak{a})}$

$\boxed{R \text{ Noetherian } + \dim 0 \implies \text{Artinian}}$

Theorem: Let R be a commutative Noetherian ring R of dim 0 (that is, every prime ideal is maximal). Then, R is Artinian.

Pf idea: Use all the tools from Lecture 23.

- Str. Theorem: $\tilde{R}$ Artinian $\Rightarrow$ • f. many mxl ideals $m_1, \dots, m_e$
  - $\exists n$: $\mathcal{N}^n = (0)$
  - $\tilde{R} \cong \tilde{R}/m_1^n \times \cdots \times \tilde{R}/m_e^n$
- $\dim_k m_j^i/m_j^{i+1} < \infty$ for all i. $(k = \tilde{R}/m_j)$

- By the Corollary, R has finitely many minimal primes $\underset{\dim 0}{\Longrightarrow}$ they are mxl! Call them $\boxed{m_1, \dots, m_e}$.
- No other max ideals:

$\Longrightarrow \{m_1, \ldots, m_\ell\} = \mathrm{MaxSpec}(R)$ (set of mxl ideals of R)

Uniqueness Properties of Primary Decomposition

⚠ Decompositions are in general **NOT** unique, but certain features will be:

Example: $\mathfrak{a} = (x^2, xy) \subseteq R = \mathbb{K}[x,y]$ ($\mathbb{K}$ = any field)

Main result:
- $\mathfrak{p}_i$'s will be unique
- After "minimizing" the decomp: $\mathfrak{q}_i$ assoc to min primes over $\mathfrak{a}$ will be unique

Reduced Prim Decomp

- We fix (*) $\boxed{\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\ell}$ primary decomp $\quad \mathfrak{p}_i = r(\mathfrak{q}_i)\ \forall i$ (prime)

STEP 1: Create a "reduced primary decomp" from (*)

Def: The decomp (*) is reduced if the following assumptions hold:

(1) $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$ are all distinct.

(2) $\mathfrak{q}_i \not\supseteq \bigcap_{\substack{1 \le j \le \ell \\ j \neq i}} \mathfrak{q}_j$ for all $i = 1, \ldots, \ell$. (ie, no $\mathfrak{q}_i$ is redundant)

- Removing redundant $\mathfrak{q}_i$'s we can assume (2) holds.
- (1) will be fulfilled by the next Lemma:

Lemma: If $\tilde{\mathfrak{q}}_1, \ldots, \tilde{\mathfrak{q}}_n$ are primary ideals with $r(\tilde{\mathfrak{q}}_i) = \mathfrak{p}$ $\forall i = 1, \ldots, n$, then $\tilde{\mathfrak{q}} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is also primary and $r(\tilde{\mathfrak{q}}) = \mathfrak{p}$

Proof: We need to show: $\tilde{q} = \bigcap_{i=1}^{n} q_i$ primary & $r(\tilde{q}) = P = r(\tilde{q}_i)$.

STEP 2: Analyze uniqueness features of reduced prim decomp.

Theorem 2 The set of prime ideals $\{P_1, \dots P_\ell\}$ is uniquely determined by $\mathfrak{a}$. More precisely:

$$\{P_1, \dots, P_\ell\} = \{ r((\mathfrak{a}:x)) \; : \; x \in R \;\&\; r((\mathfrak{a}:x)) \text{ is prime}\}$$

this does NOT require a primary decomp.
(so (LHS) is indep of our choice of red. primary decomp)

Lemma: Let $\mathfrak{q} \subsetneq R$ be primary & $\mathfrak{p} := r(\mathfrak{q})$. Given $x \in R$, we have:

(1) $x \in \mathfrak{q} \implies (\mathfrak{q} : x) = R$

(2) $x \notin \mathfrak{q} \implies (\mathfrak{q} : x)$ is primary & $r(\mathfrak{q} : x) = \mathfrak{p}$.

(3) $x \notin \mathfrak{p} \implies (\mathfrak{q} : x) = \mathfrak{q}$.

Recall: $(\mathfrak{a} : \mathfrak{b}) = \{ r \in R : r\mathfrak{b} \subseteq \mathfrak{a} \}$ for $\mathfrak{a}, \mathfrak{b}$ ideals.

Proof: