R Commutative

Lecture 25: Primary Decomposition II

Recall . $\mathfrak{q}$ primary ideal if $xy \in \mathfrak{q}$ & $y \notin \mathfrak{q} \Rightarrow x^n \in \mathfrak{q}$ for some n

. $\mathfrak{q}$ irred ideal if $\mathfrak{q} = \mathfrak{b} \cap \mathfrak{c}$ with $\mathfrak{b}, \mathfrak{c}$ ideals $\Rightarrow \mathfrak{q} = \mathfrak{b}$ or $\mathfrak{q} = \mathfrak{c}$.

Lemma: $\mathfrak{q}$ primary $\Longrightarrow r(\mathfrak{q})$ prime

Obs: 1. $\mathfrak{q}$ primary $\not\Rightarrow$ $\mathfrak{q}$ is a power of a prime

2. $\mathfrak{p}$ prime $\not\Rightarrow$ $\mathfrak{p}^n$ is primary

3. $r(\mathfrak{q})$ maximal $\Longrightarrow$ $\mathfrak{q}$ is primary

Prop: R Noetherian : (i) Every ideal is a finite intersection of irred. ideals

(ii) Irred $\Longrightarrow$ Primary.

. **Consequence**: "Primary Decomposition" for Noetherian rings.

. $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ & $\mathfrak{p}$ minimal among prime ideals containing $\mathfrak{a}$

then $\mathfrak{p} = r(\mathfrak{q}_i)$ for some i

Consequence: $\mathfrak{a}$ has finitely many minimal associate primes. $\boxed{= \operatorname{Min}(\mathfrak{a})}$

$\boxed{R \text{ Noetherian} + \dim 0 \implies \text{Artinian}}$

<u>Theorem:</u> Let R be a commutative Noetherian ring R of dim 0 (that is, every prime ideal is maximal). Then, R is Artinian.

<u>Pf idea:</u> Use all the tools from Lecture 23.

- Str. Theorem : $\tilde{R}$ Artinian $\Rightarrow$ • f. many mxl ideals $m_1, \ldots, m_\ell$
  - $\exists n : \mathcal{N}^n = (0)$
  - $\tilde{R} \simeq \tilde{R}/m_1^n \times \cdots \times \tilde{R}/m_\ell^n$
- $\dim_k m_j^i / m_j^{i+1} < \infty$ for all i. $(k = \tilde{R}/m_j)$

- By the Corollary, R has finitely many minimal primes $\underset{\dim 0}{\implies}$ they are mxl! Call them $\boxed{m_1, \ldots, m_\ell}$.
- <u>No other max ideals</u>: Pick m mxl. Show m is a min prime. If not $\exists\, \wp \subsetneq m$ $\wp$ prime $\overset{\dim 0}{\implies}$ $\wp$ mxl forces $\wp = m$ contr..

$\Rightarrow \{m_1, \dots, m_\ell\} = \mathrm{MaxSpec}(R)$ (set of mxl ideals of R)

- Claim 1: $\mathcal{N} = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p} = \bigcap_{\mathfrak{p} \text{ min prime}} \mathfrak{p} = m_1 \cap \cdots \cap m_\ell \underset{\text{pairwise coprime}}{=} m_1 \cdots m_\ell$

- Claim 2: $\exists n$ st $\mathcal{N}^n = (0)$

Pf/ $\mathcal{N}$ is fg $\mathcal{N} = (x_1, \dots, x_s)$ & $x_1^{m_1} = \dots = x_s^{m_s} = 0$ for some $m_1 \dots m_s \in \mathbb{Z}_{\geq 1}$.

Pick $k = \max\{m_1, \dots, m_s\}$ so $x_i^k = 0$ $\forall i = 1, \dots, s$.

Pick $n > s(k-1)$

Write $y_i = \sum_{j=1}^{s} a_j^{(i)} x_j \in \mathcal{N}$

$\Rightarrow y_1 \cdots y_n = (a_1^{(1)} x_1 + \dots + a_s^{(1)} x_s) \cdots (a_1^{(n)} x_1 + \dots + a_s^{(n)} x_s)$

equals 0 since after distributing, each summand must contain some x_j raised to a power $> k-1$ (ie $\geq k$). So $\mathcal{N}^n = (0)$. $\square$

- Using the claims & the proof of the Structure Theorem, we have $R \simeq R/m_1^n \times \cdots\cdots \times R/m_\ell^n$ (Use the same proof!)

- Each R/m_j^n is Noetherian, of dim 0 & local (with unique maximal ideal $\bar{m}_j = m_j/m_j^n$). If we show R/m_j^n is Art., we are done. (Prod of Art. is Art. HW9)

- The Noetherian condition says $\bar{m}_j^{\,i}/\bar{m}_{j+1}^{\,i+1}$ is an R/m_j -v.s of finite dim (it's f.g as an R-module, so f.g as an R/m_j-mod).

The proof technique of (Artinian + local $\Rightarrow$ Noetherian) works here as well. And shows R/m_j^n is Artinian

Why? strictly descending chain of ideals in R/m_j^n $\Longrightarrow$ infinite list of subspaces in $\bar{m}_j^{\,i}/m_{j+1}^{\,i+1}$ $i=1,\dots,n$. Contra!

Uniqueness Properties of Primary Decomposition

⚠ Decompositions are in general **NOT** unique, but certain features will be:

Example: $\mathfrak{a} = (x^2, xy) \subseteq R = K[x,y]$ (K = any field)

Let $\mathfrak{p}_1 = (x)$, $\mathfrak{p}_2 = (x,y)$. Both are prime ideals

$$\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2^2 = \mathfrak{p}_1 \cap (x^2, y)$$ are 2 distinct primary dec of $\mathfrak{a}$

- $\mathfrak{p}_1$ primary $fg \in (x)$ & $x \nmid g \Rightarrow x \mid f$.
- $\mathfrak{p}_2^2$ " because $r(\mathfrak{p}_2^2) = (x,y)$ is mxl.
- (x^2, y) " " $r((x^2,y)) = (x,y)$ ——.

Both have $\mathfrak{p}_1$ in common. This is no accident!

Main result: . $\mathfrak{p}_i$'s will be unique
. After "minimizing" the decomp: $\mathfrak{q}_i$ assoc to min primes over $\mathfrak{a}$ will be unique

Reduced Prim Decomp

- We fix (*) $\boxed{\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\ell}$ primary decomp $\mathfrak{p}_i = r(\mathfrak{q}_i)\ \forall i$ (prime)

STEP 1: Create a "reduced primary decomp" from (*)

Def: The decomp (*) is reduced if the following assumptions hold:

(1) $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$ are all distinct.

(2) $\mathfrak{q}_i \not\supseteq \bigcap_{\substack{1\le j\le \ell \\ j\neq i}} \mathfrak{q}_j$ for all $i = 1, \ldots, \ell$. (ie, no $\mathfrak{q}_i$ is redundant)

- Removing redundant $\mathfrak{q}_i$'s we can assume (2) holds.
- (1) will be fulfilled by the next Lemma:

Lemma: If $\tilde{\mathfrak{q}}_1, \ldots, \tilde{\mathfrak{q}}_n$ are primary ideals with $r(\tilde{\mathfrak{q}}_i) = \mathfrak{p}$ $\forall i = 1, \ldots, n$, then $\tilde{\mathfrak{q}} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is also primary and $r(\tilde{\mathfrak{q}}) = \mathfrak{p}$

Proof: We need to show: $\tilde{\mathfrak{q}} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ primary & $r(\tilde{\mathfrak{q}}) = \mathfrak{p} = r(\tilde{\mathfrak{q}}_i)$.

How? Use $r(\mathfrak{a}_1 \cap \mathfrak{a}_2) = r(\mathfrak{a}_1) \cap r(\mathfrak{a}_2)$ for all ideals $\mathfrak{a}_1, \mathfrak{a}_2$

- $r(\tilde{\mathfrak{q}}) = \bigcap_{i=1}^{n} r(\tilde{\mathfrak{q}}_i) = \bigcap_{i=1}^{n} \mathfrak{p} = \mathfrak{p}$
- Pick $xy \in \tilde{\mathfrak{q}}$ with $y \notin \tilde{\mathfrak{q}}$. Then, $y \notin \tilde{\mathfrak{q}}_j$ for some j & $xy \in \mathfrak{q}_j$

$\underset{\mathfrak{q}_j \text{ primary}}{\Longrightarrow}$ $x^n \in \mathfrak{q}_j$, i.e. $x \in r(\mathfrak{q}_j) = \mathfrak{p} = r(\tilde{\mathfrak{q}})$ so $x^N \in \tilde{\mathfrak{q}}$ for some $N > 0$.

$\square$

STEP 2: Analyze uniqueness features of reduced prim decomp.

Theorem 2 The set of prime ideals $\{\mathfrak{p}_1, \ldots \mathfrak{p}_\ell\}$ is uniquely determined by $\mathfrak{a}$. More precisely:

$$\{\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell\} = \{ r((\mathfrak{a} : x)) : x \in R \text{ \& } r((\mathfrak{a} : x)) \text{ is prime}\}$$

this does NOT require a primary decomp.

(so (LHS) is indep of our choice of red. primary decomp)

Proof: Next Time.

Lemma: Let $\mathfrak{q} \subsetneq R$ be primary & $\mathfrak{p} := r(\mathfrak{q})$. Given $x \in R$, we have:

(1) $x \in \mathfrak{q} \Rightarrow (\mathfrak{q} : x) = R$

(2) $x \notin \mathfrak{q} \Rightarrow (\mathfrak{q} : x)$ is primary & $r(\mathfrak{q} : x) = \mathfrak{p}$.

(3) $x \notin \mathfrak{p} \Rightarrow (\mathfrak{q} : x) = \mathfrak{q}$.

Recall: $(\mathfrak{a} : \mathfrak{b}) = \{ r \in R : r\mathfrak{b} \subseteq \mathfrak{a} \}$ for $\mathfrak{a}, \mathfrak{b}$ ideals.

Proof: (1) is clear,

For (3): if $y \in (\mathfrak{q} : x)$ & $x \notin \mathfrak{p}$, then $y \in \mathfrak{q}$ (otherwise,

$xy \in \mathfrak{q}$ & $y \notin \mathfrak{q} \Rightarrow x \in r(\mathfrak{q}) = \mathfrak{p}$)

So $\underset{\substack{\uparrow \\ \text{Always}}}{\mathfrak{q} \subseteq (\mathfrak{q} : x)} \subseteq \mathfrak{q}$ gives $(\mathfrak{q} : x) = \mathfrak{q}$.

(2) Next Time.