Lecture 26: Primary Decomposition (III)

Last Time: a Primary Decomp $\rightsquigarrow$ a Reduced One: $\boxed{\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\ell}$

(1) $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$ are all distinct.

(2) $\mathfrak{q}_i \not\supseteq \bigcap_{\substack{1 \leq j \leq \ell \\ j \neq i}} \mathfrak{q}_j$ for all $i = 1, \ldots, \ell$. (ie, no $\mathfrak{q}_i$ is redundant)

Example $\mathfrak{a} = (x^2, xy) = (x) \cap (x,y)^2 = (x) \cap (x^2, y)$ reduced, not unique!
$\mathfrak{p}_1 = (x)$, $\mathfrak{p}_2 = (x,y)$

TODAY: ① Discussed uniqueness features of reduced primary decomp

② Special case: R is a PID

Theorem 1: The set of prime ideals $\{\mathfrak{p}_1, \ldots \mathfrak{p}_\ell\}$ of a reduced primary decomposition of $\mathfrak{a}$ is uniquely determined by $\mathfrak{a}$. More precisely:

$$\{\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell\} = \{ r((\mathfrak{a} : x)) : x \in R \ \& \ r((\mathfrak{a}:x)) \text{ is prime}\}$$

this does NOT require a primary decomp.
(so (LHS) is indep of our choice of red. primary decomp)

Lemma: Let $\mathfrak{q} \subsetneq R$ be primary & $\mathfrak{p} := r(\mathfrak{q})$. Given $x \in R$, we have:

(1) $x \in \mathfrak{q} \implies (\mathfrak{q} : x) = R$

(2) $x \notin \mathfrak{q} \implies (\mathfrak{q} : x)$ is primary & $r(\mathfrak{q} : x) = \mathfrak{p}$.

(3) $x \notin \mathfrak{p} \implies (\mathfrak{q} : x) = \mathfrak{q}$.

Recall: $(\mathfrak{a} : \mathfrak{b}) = \{ r \in R : r\mathfrak{b} \subseteq \mathfrak{a} \}$ for $\mathfrak{a}, \mathfrak{b}$ ideals.

Proof: (1) is clear.

For (3): if $y \in (\mathfrak{q} : x)$ & $x \notin \mathfrak{p}$, then $y \in \mathfrak{q}$ (otherwise,

$xy \in \mathfrak{q}$ & $y \notin \mathfrak{q} \implies x \in r(\mathfrak{q}) = \mathfrak{p}$)

So $\mathfrak{q} \underset{\text{Always}}{\subseteq} (\mathfrak{q} : x) \subseteq \mathfrak{q}$ gives $(\mathfrak{q} : x) = \mathfrak{q}$.

For (2): To show: $x \notin q \Rightarrow (q : x)$ is primary & $r(q : x) = \wp$

<u>Theorem</u>: The set of prime ideals $\{\mathfrak{P}_1, \ldots \mathfrak{P}_\ell\}$ in a reduced decomp of $\mathfrak{a}$ is $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_\ell\} = \{ r((\mathfrak{a} : x)) : x \in R$ & $r((\mathfrak{a} : x))$ is prime$\}$

Notation $\boxed{\mathrm{Assoc}(\mathfrak{a})} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_\ell\}$ is the set of primes associated to $\mathfrak{a}$ (it doesn't depend on a prim decomp, but solely on $\mathfrak{a}$).

Corollary: If $\mathfrak{p} \subsetneq R$ prime ideal minimal among primes containing $\mathfrak{a}$, then $\mathfrak{p} \in \mathrm{Assoc}(\mathfrak{a})$. That is, $\boxed{\mathrm{Min}(\mathfrak{a}) \subseteq \mathrm{Ass}(\mathfrak{a})}$

- We relabel $\mathrm{Assoc}(\mathfrak{a})$ so that $\mathrm{Min}(\mathfrak{a}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ for some $k \leq \ell$
- Assuming $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\ell$ is reduced primary decomp, where:

Theorem: $\{\mathfrak{q}_1, \dots, \mathfrak{q}_k\}$ are uniquely determined by $\mathfrak{a}$.
More explicitly, $\mathfrak{q}_i = j_i^{-1}(j_i(\mathfrak{a}) R_{\mathfrak{p}_i})$ for $i = 1, \dots, k$,
where $j_i : R \longrightarrow R_{\mathfrak{p}_i}$.

Primary decomposition for PID's

Def: A commutative ring R is a principal ideal domain (PID) if it is a domain and every ideal of R can be generated by 1 element.

Observation: PID $\Longrightarrow$ Noetherian.

Ex: 1. $\mathbb{Z}$

2. $\mathbb{K}[x]$

Q: What do primary decompositions look like for PIDs?

Lemma: Fix R = PID & $\mathfrak{p} \subsetneq R$ nonzero prime ideal. Then, $\mathfrak{p}$ is maximal

Corollary: (HW9) All nonzero primary ideals in PID have maximal radicals.

Theorem (Prim Decomp for PIDs)

Given $(0) \neq \mathfrak{a} \subsetneq R$ ideal in a PID, there exist primary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_\ell$ s.t

(1) $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\ell$

(2) $\{ \mathfrak{p}_i = r(\mathfrak{q}_i) \}_{1 \le i \le \ell}$ are distinct non-zero prime ideals

(3) $\mathfrak{q}_i \not\supseteq \bigcap_{j \neq i} \mathfrak{q}_j$

Furthermore $\operatorname{Min}(\mathfrak{a}) = \operatorname{Ass}(\mathfrak{a})$, so we get uniqueness of all $\mathfrak{q}_1, \ldots, \mathfrak{q}_\ell$.

Q: What more can we say about primary ideals?

Lemma: If R is a PID & $(0) \neq \mathfrak{q}$ is primary, then $\mathfrak{q} = \mathfrak{m}^n$ for some $n > 0$ where $\mathfrak{m} = r(\mathfrak{q})$ is a maximal ideal.

Corollary: $\mathfrak{a} = \mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_\ell^{n_\ell} = \mathfrak{p}_1^{n_1} \cdot \cdots \cdot \mathfrak{p}_\ell^{n_\ell}$. (because $\mathfrak{p}_i$ max'l)