Lecture 27: Modules over PIDs

Recall: R PID = • all ideals can be generated by just 1 element
• R a commutative domain (no zero divisors)

Today's Goal: Classify finitely generated modules over PID's

Application: finitely generated abelian gps = fg $\mathbb{Z}$-modules
$\rightsquigarrow$ Classification for f.g ab. gps.

• Elements of a module M come in 2 flavours:
- $\mathrm{Ann}(m) = (0)$ $\rightsquigarrow$ m is a "free" element
- $\underset{\substack{\cap\mid \\ R \text{ ideal}}}{\mathrm{Ann}(m)} \neq (0)$ $\rightsquigarrow$ $\mathrm{Ann}(m) = (f)$ $f \neq 0$
  so m is a "torsion element".

$\rightsquigarrow$ M will be decomposed into a "free part" and a "torsion part".

Free modules.

Def: An R-module M is free if $M \underset{\varphi}{\xleftarrow{\simeq}} \bigoplus_{i \in I} R \quad (= R^{\oplus I})$ for some I. We say $\{\varphi(e_i) : i \in I\}$ is a basis for M.

Theorem: If R is commutative an M is a free module, then any two bases for M have the same cardinality. ($=:$ **rank** of M)

1 $\{\overline{x_i}\}_{i\in I}$ spans:

2 $\{\overline{x_i}\}_{i\in I}$ is li:

Obs: If R is a PID, then $\mathfrak{m} = (p)$ for some p. We refer to p as a prime element in R. (Recall for PID every nonzero prime ideal is mxl)

• Next, we need to ensure freeness is preserved for submodules:

This is not true in general!

Theorem 2: Let F be a free module over a PID R & M a submodule. Then, M is free and $\operatorname{rank}(M) \le \operatorname{rank}(F)$.

Torsion Modules

Def: Let M be an R-module. We say M is a <u>torsion module</u> if given $x \in M$ $\exists\, a \in R \setminus \{0\}$ with $ax = 0$ (equivalently $\mathrm{Ann}(x) \neq (0)$ $\forall x \in M$).

Obs: Finite abelian gp translates to finitely generated torsion module

Def: A <u>torsion element</u> x of a module M is $x \in M$ with $\mathrm{Ann}(x) \neq (0)$.

Write $\boxed{M_{\mathrm{tor}}} = \{$ torsion elements of $M\}$

Def If $M_{\mathrm{tor}} = \{0\}$, we say M is <u>torsion free</u>.

⚠ Torsion free + fg $\not\Rightarrow$ Free

However, the statement is true for modules over PIDs!

Prop: M a fg R-module, R a PID. If M is torsion free, then M is free.

Theorem 3: Fix R a PID and M a f.g R-module. Then, M/M_{Tor} is a free R-module. Furthermore, there exists a free submodule F of M with $M = M_{\text{Tor}} \oplus F$. & rank$(F)$ is uniquely determined by M.

Lemma: Consider M & M' two modules over a PID R with M' free.
Fix $f: M \longrightarrow M'$ a surjective homomorphism of R-modules.
Then, there exists a free submodule N of M such that
(1) $f|_N$ induces an isomorphism $f|_N : N \xrightarrow{\sim} M'$.
(2) $M = N \oplus \ker f$.

$\boxed{\text{Definitions / Notation for Classification Thm}}$

Fix R PID

Def: We say $p \in R$ is a <u>prime</u> element if $(0) \neq (p)$ is a prime ideal

<u>Notation</u>: For a module M over R, and $p \in R$ prime, we write:

$$\boxed{M_{(p)}} = \{ m \in M \mid \operatorname{Ann}(m) = (p^r) \text{ for some } r \geq 1 \}$$

p-torsion pts of M

Def: A p-submodule of M is a submodule contained in $M_{(p)}$

- We <u>select representatives</u> for the prime elements of R, modulo units

<u>Examples</u>: $\mathbb{Z}$

$\mathbb{K}[x]$

Def: Given $a \in R \setminus \{0\}$, we write $M_a = \operatorname{Ker}\left(M \xrightarrow{a\cdot} M \right)$

$x \longmapsto a \cdot x$

Def: An R-module M is <u>cyclic</u> if $M \simeq R/(a)$ for some a.

Def: A p-module M $(M = M(p))$ is of Type $(p^{r_1}, \dots, p^{r_s})$ if it is isomorphic to $\prod_{i=1}^{s} R/(p_i^{r_i})$

If p is understood, we say M has Type $(r_1, \dots, r_s)$

Classification Thm: If M is a fg Torsion module over a PID R, then:

$$M = \bigoplus_{\substack{p \text{ prime} \\ M(p) \neq 0}} M_{(p)} .$$

Furthermore: $M_{(p)} \cong R/(p^{v_1}) \oplus \dots \oplus R/(p^{v_s})$ with $1 \leq v_1 \leq \dots \leq v_s$

The sequence (v_i) is uniquely determined by M & p.

Proof: Next time.