Lecture 27: Modules over PIDs

Recall: R PID = • all ideals can be generated by just 1 element
• R a commutative domain (no zero divisors)

Today's Goal: Classify finitely generated modules over PID's

Application: finitely generated abelian gps = fg $\mathbb{Z}$-modules
$\rightsquigarrow$ Classification for f.g ab. gps.

• Elements of a module M come in 2 flavours:

• $\mathrm{Ann}(m) = (0)$ $\rightsquigarrow$ m is a "free" element
• $\underset{\substack{\cap\!| \\ R}}{\mathrm{Ann}(m)} \neq (0)$ ideal $\rightsquigarrow$ $\mathrm{Ann}(m) = (f)$ $f \neq 0$
so m is a "torsion element".

$\rightsquigarrow$ M will be decomposed into a "free part" and a "torsion part".

Free modules

Def: An R-module M is **free** if $M \underset{\varphi}{\xleftarrow{\simeq}} \bigoplus_{i \in I} R \quad (= R^{\oplus I})$ for some I. We say $\{\varphi(e_i) : i \in I\}$ is a basis for M.

Theorem: If R is commutative an M is a free module, then any two bases for M have the same cardinality. $(=:$ **rank** of $M)$

Pf/ Consider $\mathfrak{m}$ a maximal ideal on R. Then $\overline{M} = M/\mathfrak{m}M$ is a k-v.sp for $k = R/\mathfrak{m}$. $\Rightarrow$ $\overline{M}$ has a basis & all bases have the same cardinality.

Furthermore: If $(x_i)_{i \in I}$ is a basis for M, then $\overline{x_i} = x_i + \mathfrak{m}M$ gives a a basis for $\overline{M}$ (so $|I| = \dim_k \overline{M}$ does not depend on the basis)

Indeed we show $\{\overline{x_i}\}_{i \in I}$ both spans $\overline{M}$ & is li:

(LI: $0 = \sum_{\substack{i \in I \\ \text{finite}}} a_i \overline{x_i} \Rightarrow a_i = 0$ for this finite set)

① $\{\overline{x_i}\}_{i\in I}$ spans:

If $\overline{x} \in M/mM$, then $x = \sum_{\substack{i\in I\\ \text{finite}}} a_i \overline{x_i}$ with $a_i \in R$

So $\overline{x} = \sum_{\substack{i\in I\\ \text{finite}}} (a_i + m)\overline{x_i}$

② $\{\overline{x_i}\}_{i\in I}$ is li: If $\overline{0} = \sum_{\substack{i\in I\\ \text{finite}}} (a_i + m)\overline{x_i}$, then $\sum_{\substack{i\in I\\ \text{finite}}} a_i x_i \in mM$,

so $x = \sum_{\substack{i\in I\\ \text{finite}}} a_i x_i = \sum_{j=1}^{m} \underset{\substack{\cap\\ m}}{b_j} \cdot \overset{\in M}{y_j}$ with $y_j = \sum_{\substack{i\in I\\ \text{finite}}} c_i^{(j)} x_i$

$\Rightarrow \sum_{j=1}^{m} b_j \sum_{\substack{i\in I\\ \text{finite}}} c_i^{(j)} x_i \underset{\text{Assoc}}{=} \sum_{\substack{i\in I\\ \text{finite}}} \underbrace{\left(\sum_{j=1}^{m} b_j c_i^{(j)}\right)}_{\in m} x_i$

By definition of $\bigoplus_{i\in I} R$, $a_i \in m$ $\forall i$ in supp of x, $a_i = 0$ otherwise.

So $a_i + m = 0$ $\forall i$. We conclude $\{\overline{x_i}\}$ is l.i. $\square$

Obs: If R is a PID, then $\mathfrak{m} = (p)$ for some p. We refer to p as a **prime** element in R. (Recall for PID every nonzero prime ideal is mxl)

- Next, we need to ensure freeness is preserved for submodules:

This is not true in general!

Example: $R = K[x,y]$ $\quad M = R$ is free of rank 1 but $I = (x,y)$ is not a free submodule of M.

- $I \not\cong R$ (not a cyclic module)
- We have the obvious relation: $y\underline{\underline{x}} - x\underline{\underline{y}} = 0$ in I
- Any $\{f_i\}_{i \in I}$ generating set will have obvious rels $f_i \underbrace{f_j}_{\text{in } I} - f_j \underbrace{f_i}_{\text{in } I} = 0$.

Theorem 2: Let F be a free module over a PID R & M a submodule. Then, M is free and $\operatorname{rank}(M) \leq \operatorname{rank}(F)$.

Proof: We discuss the finite case (For the case when rank(F) is infinite, see HW10). Assume F has a basis $\{x_i\}_{i=1}^{n}$ $(n = \text{rank} F)$.

Let $M_r = M \cap (x_1, \dots, x_r)$ for $r = 1, \dots, n$.

We show M_r is free of rank $\leq r$ by induction on r:

- **Base case**: $r = 1$ $\quad M_1 = M \cap (x_1)$ is a submodule of (x_1), so

$M_1 = (a_1 x_1)$ for some $a_i \in R$. $\Longrightarrow$ $M_1 = 0$ or $M_1 \simeq R$ $\quad$ $\left(\text{Ann}(x_1) = 0\right)$ R domain

- **Inductive step**: Let $\mathfrak{a} = \{a \in R : \exists\ x \in M$ with $x = b_1 x_1 + \dots + b_r x_r + a x_{r+1}\}$ for $b_1, \dots, b_r \in R$

Claim: $\mathfrak{a}$ is an ideal (because M is an R-module) $\underset{R \text{ PID}}{\Longrightarrow}$ $\mathfrak{a} = (a_{r+1}) \leq R$

(1) If $a_{r+1} = 0$, then $M_{r+1} = M_r$ so M_{r+1} is free of rank $\leq r$.

(2) If $a_{r+1} \neq 0$, we pick $w \in M_{r+1}$ with $w = \underbrace{b_1 x_1 + \dots + b_r x_r}_{\in (x_1, \dots x_r)} + a_{r+1} x_{r+1}$ $\quad$ (*) $a_{r+1} \neq 0$, so $\text{Ann}(w) = (0)$

For any $x \in M_{r+1}$ we write $x = a_1 x_1 + \dots + a_r x_r + \underbrace{(c\, a_{r+1})}_{\in \mathfrak{a}} x_{r+1}$

so $x - cw \in M \cap (x_1, \dots, x_r)$

$s \leq r$

$R^s \quad R$

$\text{\rotatebox{90}{$\cong$}} \quad \text{\rotatebox{90}{$\cong$}}$ (*)

$\Longrightarrow$ $\boxed{M_{r+1} = M_r + (w)}$ & $M_r \cap (w) = (0)$ $\Longrightarrow$ $M_{r+1} = M_r \oplus (w) \simeq R^{s+1}$ $\square$

Torsion Modules

Def: Let M be an R-module. We say M is a <u>torsion module</u> if given $x \in M$ $\exists\, a \in R \setminus \{0\}$ with $ax = 0$ (equivalently $\mathrm{Ann}(x) \neq (0)$ $\forall x \in M$).

Obs: Finite abelian gp translates to finitely generated torsion module

Def: A <u>torsion element</u> x of a module M is $x \in M$ with $\mathrm{Ann}(x) \neq (0)$.

Write $\boxed{M_{tor}} = \{$torsion elements of $M\}$

Def If $M_{tor} = \{0\}$, we say M is <u>torsion free</u>.

⚠ Torsion free + f.g. $\not\Rightarrow$ Free

Ex: $M = (x, y)$ torsion free $\mathbb{K}[x,y]$-mod, but NOT free.

However, the statement is true for modules over PIDs!

Prop: M a fg R-module, R a PID. If M is torsion free, then M is free.

Pf/ Pick $\mathcal{Y} = \{y_1, \dots, y_m\}$ generators for M. Let $S = \{v_1, \dots, v_n\}$ a maximal li subset in $\mathcal{Y}$

- Pick $y \in \mathcal{Y} \setminus S$. Then: $\exists\, a, b_1, \dots, b_n \in R$, not all 0, so

$$ay + b_1 v_1 + \dots + b_n v_n = 0. \quad \underset{S \text{ li}}{\Longrightarrow} \quad a \neq 0$$

- If $y \in S$, then $y = v_i$ & $\overset{a}{1} \cdot y - 1 \cdot v_i = 0$.

$\Rightarrow$ $\forall j = 1, \dots, m$ $\exists\, a_j \neq 0$ with $a_j y_j \in (v_1, \dots, v_n)$. Take $\boxed{a = a_1 \cdots a_m}$

Then $aM \subseteq (v_1 \dots v_n)$ & $a \neq 0$. (R domain)

$\rightsquigarrow$ $\varphi_a : M \longrightarrow M$

$\qquad m \longmapsto a \cdot m$

$$\left.\begin{array}{l} \cdot\ \varphi_a \text{ inj} \\ \cdot\ \operatorname{Im} \varphi_a \subseteq N = (v_1, \dots v_n) \\ \cdot\ N = (v_1 \dots v_n) \text{ free (has a basis)} \\ \quad \operatorname{rank}(N) \leq n \end{array}\right\} \Rightarrow M \simeq \varphi_a(M) \text{ also free of } \operatorname{rank} \leq \operatorname{rank} N = n \leq m$$

$\square$

Theorem 3: Fix R a PID and M a f.g R-module. Then, M/M_{tor} is a free R-module. Furthermore, there exists a free submodule F of M with $M = M_{tor} \oplus F$. & rank(F) is uniquely determined by M.

Proof: • **Claim 1**: $\overline{M} = M/M_{tor}$ is torsion free.

Pf/

Let $\overline{x} \in \overline{M}$ and $b \in R$ with $b\overline{x} = 0$ in $\overline{M}$. Then, $bx \in M_{tor}$ so $\mathrm{Ann}(bx) \neq (0)$.

But $\mathrm{Ann}(bx) = (c)$ $c \neq 0$ gives $(bc)x = 0$ in M.

So either $bc = 0$ or $x \in M_{tor}$ $(\Rightarrow \overline{x} = 0)$

$\Downarrow c \neq 0$

$b = 0$. & $\overline{x}$ is not a torsion element of $\overline{M}$

• M is fg, so $\overline{M}$ is fg

• $\overline{M}$ is fg & torsion free. By Proposition, it is free as an R-module. It's rank is uniquely determined by M. To finish, build $F \subseteq M$ free with $F \simeq M/M_{tor}$

• To find F, we need a lemma applied to $\varphi : M \twoheadrightarrow M/M_{tor}$

Lemma: Consider M & M' two modules over a PID R with M' free. Fix $f: M \to M'$ a surjective homomorphism of R-modules. Then, there exists a free submodule N of M such that

(1) $f|_N$ induces an isomorphism $f|_N : N \xrightarrow{\sim} M'$.

(2) $M = N \oplus \operatorname{Ker} f$.

Proof: Pick a basis $B = \{x_i'\}_{i \in I}$ for M'. For each i, let $x_i \in M$ with $f(x_i) = x_i'$.

Take $N = (x_i : i \in I)$

Claim: $\{x_i : i \in I\}$ is l.i.

Pf/ $\sum_{\substack{i \in I \\ \text{finite}}} a_i x_i = 0 \rightsquigarrow \sum_{\substack{i \in I \\ \text{finite}}} a_i \underbrace{f(x_i)}_{= x_i'} = 0 \underset{B \text{ basis}}{\Longrightarrow} a_i = 0 \ \forall i$.

Conclusion: N is free with basis $\{x_i\}_{i \in I}$. & $f|_N : N \twoheadrightarrow M'$

For $x \in M$ write $f(x) = \sum_{\substack{i \in I \\ \text{finite}}} a_i f(x_i) \Rightarrow x - \sum_{\substack{i \in I \\ \text{finite}}} a_i x_i \in \operatorname{Ker} f$

. $\operatorname{Ker} f \cap N = (0)$. So $f|_N$ is iso & $M = N \oplus \operatorname{Ker} f$. $\square$

Definitions / Notation for Classification Thm

Fix R PID

Def: We say $p \in R$ is a *prime element* if $(0) \neq (p)$ is a prime ideal

Notation: For a module M over R, and $p \in R$ prime, we write:

$$\boxed{M_{(p)}} = \{ m \in M \mid \mathrm{Ann}(m) = (p^r) \text{ for some } r \geq 1 \}$$ p-torsion pts of M

Def: A p-submodule of M is a submodule contained in $M_{(p)}$

- We *select representatives* for the prime elements of R, modulo units

Examples: $\mathbb{Z} \rightsquigarrow$ positive prime numbers

$\mathbb{K}[x] \rightsquigarrow$ monic irreducible polynomials ($LT(f) = 1$)

Def: Given $a \in R \setminus \{0\}$, we write $M_a = \mathrm{Ker}\left(\begin{array}{c} M \xrightarrow{a\cdot} M \\ x \longmapsto a \cdot x \end{array} \right)$

Def: An R-module M is *cyclic* if $M \simeq R/(a)$ for some a.

Def: A p-module M $(M = M(p))$ is of type $(p^{r_1}, \ldots, p^{r_s})$ if it is isomorphic to $\prod_{i=1}^{s} R/(p_i^{r_i})$

If p is understood, we say M has type $(r_1, \ldots, r_s)$

Classification Thm: If M is a f.g. torsion module over a PID R, then:

$$M = \bigoplus_{\substack{p \text{ prime} \\ M(p) \neq 0}} M(p).$$

Furthermore: $M(p) \cong R/(p^{v_1}) \oplus \cdots \oplus R/(p^{v_s})$ with $1 \leq v_1 \leq \cdots \leq v_s$

The sequence (v_i) is uniquely determined by M & p.

Proof: Next time.