Lecture 28 : Modules over PID's II

Recall: . Last time we talked about free modules over a PID R

$M \cong \bigoplus_{i\in I} R$ via a basis $\{e_i\}_{i\in I}$ (generates + LI/R)

. Defined Torsion elements : $x \in M$ with $\mathrm{Ann}(x) \neq (0)$

$M_{Tor} = \{$torsion elements$\}$ submodule of M

. M is Torsion free module $\Leftrightarrow M_{Tor} = \{0\}$

Theorem 1: Size of the basis of a free module is unique ($=$ rank(M))

Theorem 2 : F free module over PID & M submodule, then

M is free & $\mathrm{rank}(M) \leq \mathrm{rank}(F)$

. Torsion free + fg $\not\Rightarrow$ free for general R

Proposition: M is fg over a PID & M is torsion free $\Rightarrow$ free.

Theorem 3: Fix R a PID and M a f.g R-module. Then, M/M_{Tor} is a free R-module. Furthermore, there exists a free submodule F of M with $M = M_{Tor} \oplus F$. & rank(F) is uniquely determined by M.

Proof: • **Claim 1**: $\bar{M} = M/M_{Tor}$ is torsion free.

- M is fg, so $\bar{M}$ is fg
- $\bar{M}$ is fg & Torsion free.

Lemma: Consider M & M' two modules over a PID R with M' free.
Fix $f: M \longrightarrow M'$ a surjective homomorphism of R-modules.
Then, there exists a free submodule N of M such that
(1) $f|_N$ induces an isomorphism $f|_N : N \xrightarrow{\sim} M'$.

(2) $M = N \oplus \ker f$.

Proof:

Definitions / Notation for Classification Thm

Fix R PID

Def: We say $p \in R$ is a <u>prime element</u> if $(0) \neq (p)$ is a prime ideal

Notation: Given $a \in R \setminus \{0\}$, we write $\boxed{M_a} = \operatorname{Ker}(M \xrightarrow{a\cdot} M)$
$x \longmapsto a \cdot x$

We refer to a as an <u>exponent</u> for M

Def: An R-module M is <u>cyclic</u> if $M \simeq R/(a)$ for some a.

Recall: given $a \neq 0$ we can write $\boxed{a = u p_1^{n_1} \cdots p_s^{n_s}}$ $\left(\begin{array}{l} u \in R^\times \\ p_i \text{ primes} \quad n_i \geq 1 \end{array}\right)$
where $(a) = (p_1^{n_1}) \cap \cdots \cap (p^{n_s})$ is the <u>unique</u> prim decomp (R is PID)

- We <u>select representatives</u> for the prime elements of R, modulo units
  $\rightsquigarrow$ factorization of any $a \in R \setminus \{0\}$ is unique!

Examples: $\mathbb{Z}$ $\rightsquigarrow$ positive prime numbers

$\mathbb{K}[x]$ $\rightsquigarrow$ monic irreducible polynomials
$(LT(f) = 1)$

Def: A <u>p-Torsion element</u> is any $x \in M$ with $x \in M_{p^n}$ for some $n \geq 1$.

Def: A fg p-module M ($M = M_{p^n}$ for some n) has type $(p^{r_1}, \ldots, p^{r_s})$ if it is isomorphic to $\bigoplus_{i=1}^{s} R/(p_i^{r_i})$

If p is understood, we say M has type $(r_1, \ldots, r_s)$

Classification Thm: If M is a fg torsion module over a PID R, then:

$$M = \bigoplus_{p_i \text{ prime}} M_{p_i^{n_i}} \quad \text{for } n_i \geq 1 \text{ (unique choice)}$$

Furthermore $(p = p_i)$: $M_{p^n} \cong R/(p^{\nu_1}) \oplus \cdots \oplus R/(p^{\nu_s})$ with $n = \nu_1 \geq \nu_2 \geq \cdots \geq \nu_s$

The sequence (ν_i) is uniquely determined by M & p.
(both s & ν_i' are unique).

PART 1: $M = \bigoplus_{j=1}^{s} M_{p_{ij}^{n_{ij}}}$ $n_{ij} \geq 1$ p_{ij} prime elts in R

Claim 0: M has an exponent $a \neq 0$ with $\boxed{\mathrm{Ann}(M) = (a)}$ $a \notin R^{\times}$

• Assume $a = bc$ with $(b,c) = 1$. Pick x, y with $1 = xb + yc$

Claim 1: $M = M_b \oplus M_c$

• We factor a as: $a = u p_{i_1}^{n_{i_1}} \cdots p_{i_r}^{n_{r}}$ $u \in R^{\times}$

(via Primary decomp of (a)). p_{ij} primes $n_{ij} > 0$.

CASE 1: $r = 1$

CASE 2: $r > 1$