Lecture 29: Modules over PIDs III

Last Time: M fg module over PID: $M = M_{tor} \oplus F$ F free $(\simeq M/M_{tor})$ (rank dep only on M)

- $M_a = \ker\left(M \xrightarrow{a} M\right)$ $\quad m \longmapsto a\cdot m$ $M_{tor} = \{x : \mathrm{Ann}(x) \neq (0)\}$

- p-torsion elements $= \{x \in M : \mathrm{Ann}(x) = (p^k)$ for some $k \geq 0\}$.

- **Classification Thm**: If $M \neq 0$ is a fg torsion module over a PID R, then

$$(*) \quad M = \bigoplus_{p_i \text{ prime}} M_{p_i^{n_i}} \quad \text{for } n_i \geq 1 \text{ (unique choice)}$$

Furthermore $(p = p_i)$: $M_{p^n} \cong R/(p^{\nu_1}) \oplus \cdots \oplus R/(p^{\nu_s})$ with $n = \nu_1 \geq \nu_2 \geq \cdots \geq \nu_s$

The sequence (ν_i) is uniquely determined by M & p. (Type of M_{p^n})

Pf/ Part (I): $M = M_a$ for $\mathrm{Ann}(M) = (a)$ $a \neq 0$, $a \notin R^\times$.

$\Rightarrow a = u\, p_{i_1}^{n_{i_1}} \cdots p_{i_r}^{n_{i_r}}$ (p_{i_j}) prime reps in R, $n_{i_j} \geq 1$, $u \in R^\times$

$\rightsquigarrow \boxed{M = M_{p_{i_1}^{n_{i_1}}} \oplus \cdots \oplus M_{p_{i_r}^{n_{i_r}}}}$ & uniqueness $\Longleftrightarrow$ uniqueness of prime dec. for PIDs.

PART II : $\exists$ Decomp for p-Torsion M_{p^n} $\qquad M = M_{p_{i_1}^{n_{i_1}}} \oplus \cdots \oplus M_{p_{i_r}^{n_{i_r}}}$

- Assume $M = M_{p^n}$ with n minimal.

Lemma: Assume $\operatorname{Ann}(M) = (p^n)$ & pick $x \in M$ with $\operatorname{Ann}(x) = (p^n)$. Consider the ses

$$0 \longrightarrow (x) \longrightarrow M \xrightarrow{\pi} N \longrightarrow 0$$

$$N = M/(x)$$

Then: (1) $\dim_k N/pN < \dim_k M/pM$ & (2) π admits a section (assuming N decomposes)

(2)
$$\begin{tikzcd} 0 \arrow[r] & (x) \arrow[r] & M \arrow[r, "\pi"] & N \arrow[l, dashed, bend left, "\exists s?"] = M/(x) \arrow[r] & 0 \end{tikzcd}$$
with $N = \bigoplus_{i=1}^{s} R(\overline{y_i})$,

where $R(\overline{y_i}) \simeq R/(p^{\nu_i})$ & $\nu_1 \geq \nu_2 \geq \dots \geq \nu_s \geq 1$.

$$\mathrm{Ann}(\overline{y_i}) = (p^{\nu_i})$$

We want to lift each $\overline{y_i}$ to M so that
$$\begin{cases} ① \ \mathrm{Ann}(y_i) = \mathrm{Ann}(\overline{y_i}) \\ ② \ \pi(y_i) = \overline{y_i} \end{cases}$$

- We <u>do it</u> for <u>one</u> $\overline{y} \in N \setminus \{0\}$.

- $\mathrm{Ann}(y') = \mathrm{Ann}(y - p^{s-l} c x) = \mathrm{Ann}(\bar{y})$ because

- To finish, we show $M' = R(y_1, \dots, y_s) = R(y_1) \oplus \dots \oplus R(y_s)$