Lecture 29: Modules over PIDs III

Last Time: M fg module over PID: $M = M_{tor} \oplus F$ F free ($\simeq M/M_{tor}$) (rank dep only on M)

- $M_a = \ker\left(M \xrightarrow{a} M\right)$, $m \longmapsto a\cdot m$ $M_{tor} = \{x : \mathrm{Ann}(x) \neq (0)\}$

- p-torsion elements $= \{x \in M : \mathrm{Ann}(x) = (p^k)$ for some $k \geq 0\}$.

- Classification Thm: If $M \neq 0$ is a fg torsion module over a PID R, then

$$(*) \quad M = \bigoplus_{p_i \text{ prime}} M_{p_i^{n_i}} \quad \text{for } n_i \geq 1 \text{ (unique choice)}$$

Furthermore ($p = p_i$): $M_{p^n} \cong R/(p^{\nu_1}) \oplus \cdots \oplus R/(p^{\nu_s})$ with $n = \nu_1 \geq \nu_2 \geq \cdots \geq \nu_s$

The sequence (ν_i) is uniquely determined by M & p. (Type of M_{p^n})

Pf/ Part (I): $M = M_a$ for $\mathrm{Ann}(M) = (a)$ $a \neq 0$, $a \notin R^\times$.

$\Longrightarrow a = u\, p_{i_1}^{n_{i_1}} \cdots p_{i_r}^{n_{i_r}}$ (p_{i_j}) prime reps in R, $n_{i_j} \geq 1$, $u \in R^\times$

$\rightsquigarrow \boxed{M = M_{p_{i_1}^{n_{i_1}}} \oplus \cdots \oplus M_{p_{i_r}^{n_{i_r}}}}$ & uniqueness $\Longleftrightarrow$ uniqueness of prime dec. for PIDs.

PART II: $\exists$ Decomp for p-Torsion M_{p^n}

$$M = M_{p_{i_1}^{n_{i_1}}} \oplus \cdots \oplus M_{p_{i_r}^{n_{i_r}}}$$

$\uparrow$ $p = p_{ij}$ $\quad n = n_{ij}$

- Assume $M = M_{p^n}$ with n minimal. Notice $\overline{M} = M/pM$ is a k-vsp with $k = R/(p)$ (PID's: prime & $\neq 0 \Rightarrow$ mxl)
- Since M is fg: $\dim_k \overline{M} < \infty$ $\rightsquigarrow$ induct on $\dim_k \overline{M}$
- For inductive step we need:

Lemma: Assume $\mathrm{Ann}(M) = (p^n)$ & pick $x \in M$ with $\mathrm{Ann}(x) = (p^n)$. Consider the ses

$$0 \longrightarrow (x) \longrightarrow M \xrightarrow{\pi} N \longrightarrow 0$$
$$N \simeq M/(x)$$

Then: (1) $\dim_k N/pN < \dim_k M/pM$ & (2) π admits a section (assuming N decomposes)

Pf/ (1)

$$\begin{tikzcd} M \arrow[r, two heads, "\pi_1"] \arrow[rr, bend right, "f"'] \arrow[d] & N \arrow[r, two heads, "\pi_2"] & N/pN \\ M/pM \arrow[urr, "\overline{f}"'] & & \end{tikzcd}$$

$\pi_1(pM) = p\pi_1(M) = pN$ so $pM \subseteq \ker f \Rightarrow \exists \overline{f}$

- $\overline{f}$ k-linear surjective $\rightsquigarrow$ $\dim_k M/pM \geq \dim_k N/pN$
- $(x) \in M$ satisfies $\overline{f}(x + pM) = 0$. & $x + pM \neq pM$ because $p^{n-1} \cdot x \neq 0$
- We get $\underset{k \cong}{R/(p)} \cong \dfrac{x}{p(x)} \hookrightarrow \dfrac{M}{pM} \xrightarrow{\overline{f}} \dfrac{N}{pN}$ & $\overline{f}|_{\frac{x}{p(x)}} = 0$. So Rank-Nullity gives $\dim_k M/pM > \dim_k N/pN$.

(2)
$$\begin{tikzcd} 0 \arrow[r] & (x) \arrow[r] & M \arrow[r, "\pi"] & N \arrow[l, dashed, bend left, "\exists s?"] = M/(x) \arrow[r] & 0 \end{tikzcd}$$
with $N = \bigoplus_{i=1}^{s} R(\overline{y_i})$,

where $R(\overline{y_i}) \simeq R/(p^{\nu_i})$ & $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_s \geq 1$.

$\mathrm{Ann}(\overline{y_i}) = (p^{\nu_i})$

We want to lift each $\overline{y_i}$ to M so that $\begin{cases} ① \ \mathrm{Ann}(y_i) = \mathrm{Ann}(\overline{y_i}) \\ ② \ \pi(y_i) = \overline{y_i} \end{cases}$

- We do it for one $\overline{y} \in N \setminus \{0\}$.

Assume $\mathrm{Ann}(\overline{y}) = (p^{l})$ for some $l \geq 1$. Pick an $y \in M$ with $\pi(y) = \overline{y}$.

Then $p^l y \in R(x)$. Write $p^l y = bx$ for $b \in R$ & factor b as

$$b = p^s c \quad \text{with } p \nmid c \ \& \ s \geq 0.$$

- Since $p^n x = 0$ we may assume $\boxed{s \leq n}$ (otherwise, $p^s c x = 0 = p^n c x$)
- If $s = n$, then $\left.\begin{array}{l} p^l y = 0 \\ p^{l-1} y \notin R(x) \ \text{ so } p^{l-1} y \neq 0 \end{array}\right\} \ \mathrm{Ann}(y) = \mathrm{Ann}(\overline{y})$
- If $s < n$, then $\mathrm{Ann}(p^s c x) = (p^{n-s})$ so $\mathrm{Ann}(y) = (p^{l+n-s})$

Since $p^n y = 0$ we get $l + n - s \leq n$, i.e. $\boxed{l \leq s}$

So $\boxed{y' = y - p^{s-l} c x}$ satisfies $\mathrm{Ann}(y') = (p^l)$ & $\pi(y') = \overline{y}$.

• $\mathrm{Ann}(y') = \mathrm{Ann}(y - p^{s-l}cx) = \mathrm{Ann}(\bar{y})$ because
$p^l y' = p^l y - p^s cx = 0$ & $p^{l-1}y' = p^{l-1}y - p^{s-1}cx = 0$ forces $p^{l-1}y \in R(x)$ contr.! $[\mathrm{Ann}(\bar{y}) = (p^l)]$

Assume we've lifted $\bar{y}_1, \dots, \bar{y}_s$ to $y_1, \dots, y_s$ with
• $\mathrm{Ann}(y_i) = \mathrm{Ann}(\bar{y}_i)$ & $y_i + R(x) = \bar{y}_i \in N$

Then
$$\begin{aligned} N &\xrightarrow{\ s\ } M' \\ y_i &\mapsto y_i \end{aligned}$$
section where $M' = R(y_1 \dots y_s)$

$\Rightarrow$ $M = R(x) \oplus M'$ since
• $M' \cap R(x) = \{0\}$
• $M/R(x) = M' \underset{s}{\overset{\sim}{\leftarrow}} N$ □

End of the proof of Classification Thm. (existence of the decomposition) $M = M_{p^n}$

Base case: $\dim_k M/pM = 1 = \dim_k (x)/p(x)$ forces $M = (x)$ so $M \simeq R/(p^n)$.

Inductive step: Assume N admits a decomp. since $\dim_k N/pN < \dim_k M/pM$.

Use Lemma: $M_{p^n} = \underset{\simeq R/(p^n)}{R(x)} \oplus M'$ & $M' = \bigoplus_{i=1}^{s} R(y_i) \simeq \bigoplus_{i=1}^{s} R/(p_i^{\nu_i})$
$\nu_1 \geq \dots \geq \nu_s \geq 1$

$\mathrm{Ann}(M') = (p^{\nu_1}) \geq \mathrm{Ann}(M) = (p^n)$ so $n \geq \nu_1 \geq \dots \geq \nu_s$

· To finish, we show $M' = R(y_1, \dots, y_s) = R(y_1) \oplus \cdots \oplus R(y_s)$

Indeed if $a_1 y_1 + \cdots + a_s y_s = 0$, we want to show $a_i y_i = 0 \ \forall i$

Viewed in N: $a_1 \bar{y}_1 + \cdots + a_s \bar{y}_s = 0$ in $N = \bigoplus_{i=1}^{s} R(\bar{y}_i)$ forces

$a_i \bar{y}_i = 0$ so $a_i \in \mathrm{Ann}(\bar{y}_i) = \mathrm{Ann}(y_i) \implies a_i y_i = 0$

Thus $M_{p^n} = \underset{\cong\, R/(p^n)}{(x)} \oplus \underset{\cong\, R/(p^{\nu_1})}{R(y_1)} \oplus \cdots \oplus \underset{\cong\, R/(p^{\nu_s})}{R(y_s)}$ $\quad n \geq \nu_1 \geq \cdots \geq \nu_s \geq 1$

$\square$

Uniqueness (Sketch) For M_{p^n}

$\mathrm{Ann}(M_{p^n}) = (p^n)$

$n =$ exponent of $\mathrm{Ann}(M_{p^n}) = (p^n)$ $\quad \mathrm{Ann}(x) = (p^n)$

$\nu_1 =$ ———— $\left(M_{p^n}/(x)\right) = (p^{\nu_1})$

$\nu_2 =$ ———— $\left(M_{p^n}/(x, y_1)\right) = (p^{\nu_2})$

$\vdots$

Assume $M_{p^n} = (x') \oplus \bigoplus_{i=1}^{s'} R(y_i')$ $\quad \underset{= n'}{\mathrm{ord}(x')} \geq \underset{\nu_1'}{\mathrm{ord}(y_1')} \geq \cdots \geq \underset{= \nu_{s'}'}{\mathrm{ord}(y_{s'}')}$

$\implies \mathrm{Ann}(x') = \mathrm{Ann}(x) = \mathrm{Ann}(M)$ gives $n = n'$

Claim 1: $\dim_k M/pM = s+1$ (so $1+s = \#$ summands is unique!)

Pf/ Use $M = R(x) \oplus R(y_1) \oplus \cdots \oplus R(y_s)$

$$pM = pR(x) \oplus pR(y_1) \oplus \cdots \oplus pR(y_t)$$

with $\nu_{t+1} = \cdots = \nu_s = 1$. (could have $t=s$)

So
$$\frac{M}{pM} \underset{\text{as } k \text{ v.s.}}{\cong} \underbrace{\frac{R(x)}{pR(x)}}_{\cong k} \oplus \underbrace{\frac{R(y_1)}{pR(y_1)}}_{\cong k} \oplus \cdots \oplus \underbrace{\frac{R(y_t)}{pR(y_t)}}_{\cong k} \oplus \underbrace{\bigoplus_{j=t+1}^{s} R(y_j)}_{\cong k}$$

$$\Rightarrow \dim_k \frac{M}{pM} = 1+t+(s-t) = s+1.$$

Claim 2 $\nu_i = \nu_i'$ by induction on n $\quad M = \bigoplus_{i=0}^{s} R(y_i) = \bigoplus_{i=0}^{s} R(y_i')$

Base case, $n=1$ $\quad \mathrm{Ann}(M) = (p)$ so all $\nu_i' = \nu_i = 1$

Ind step : $pM = \bigoplus_{i=0}^{t} R(py_i) \qquad \nu_{t+1} = \cdots = \nu_s = 1 \qquad \mathrm{Ann}(pM) = (p^{n-1})$

$\quad = \bigoplus_{i=0}^{t} R(py_i') \qquad \nu'_{t'+1} = \cdots = \nu'_{s'} = 1 \qquad \Rightarrow \text{IH} \quad t = t'$

$\mathrm{Ann}(py_i) = (p^{\nu_i - 1}) \qquad \mathrm{Ann}(py_i') = (p^{\nu_i' - 1}) \Longrightarrow \nu_i - 1 = \nu_i' - 1$ for $i \le t$ $\qquad \square$