Lecture 30: Modules over PIDs IV – Canonical forms for matrices

Recall:

- $M_a = \ker\left(M \xrightarrow{a} M\right)$ $\quad m \longmapsto a\cdot m$ $\qquad M_{\mathrm{tor}} = \{x : \mathrm{Ann}(x) \neq (0)\}$
- p-Torsion elements $= \{x \in M : \mathrm{Ann}(x) = (p^k)$ for some $k \geq 0\}$.
- Classification Thm: If $M \neq 0$ is a fg Torsion module over a PID R, then

$$(*) \quad M = \bigoplus_{p_i \text{ prime}} M_{p_i^{n_i}} \quad \text{for } n_i \geq 1 \text{ (unique choice)}$$

Furthermore $(p = p_i)$: $M_{p^n} \cong R/(p^{\nu_1}) \oplus \cdots \oplus R/(p^{\nu_s})$ with $n = \nu_1 \geq \nu_2 \geq \cdots \geq \nu_s$

The sequence (ν_i) is uniquely determined by M & p. (Type of M_{p^n})

TODAY:
- 2^{nd} Classification Thm
- 2^{nd} Structure Thm
- Smith Normal Forms of $m \times n$ matrices over PIDs

Classification Thm v2: If $0 \neq M$ is a f.g. torsion module over a PID R, then

$$M \cong R/(q_1) \oplus \cdots \oplus R/(q_r)$$

where $q_i \neq 0$, $q_i \notin R^{\times}$ $\forall i$ & $q_r \mid q_{r-1} \mid \cdots \mid q_1$.

Furthermore, the sequence of ideals $(q_1), \ldots, (q_r)$ is uniquely determined by the above conditions.

Claim

$$R/(p_1^{\nu_i^{(1)}}) \oplus \cdots \oplus R/(p_r^{\nu_i^{(r)}}) \simeq R/(q_i)$$

Structure Theorem

- Recall the following statement from Lecture 28:

<u>Lemma</u>: Consider M & M' two modules over a PID R.
Assume M' is free & let $f: M \longrightarrow M'$ be a surjective homomorphism of R-modules. Then, there exists a free submodule N of M such that

(1) $f|_N$ induces an isomorphism $f|_N : N \xrightarrow{\sim} M'$.

(2) $M = N \oplus \ker f$.

$(N = (x_i : i \in I)$ $\quad f(x_i) = x_i'$ $\quad (\{x_i'\}_{i \in I}$ basis for $M')$

- We'll use this to prove the following statement:

<u>Structure Thm</u> Assume R is a PID and M = f.g. free R-mod of rank n.
Fix $0 \neq N \subseteq M$ submodule. Then $\exists$ basis $\{e_1, \dots, e_n\}$ of M and $a_1, \dots, a_r \in R \setminus \{0\}$ such that

(1) $a_1 | a_2 | \cdots | a_r$.

(2) $\{a_1 e_1, \dots, a_r e_r\}$ is a basis for N

3F/ $0 \neq N \subseteq M \cong R^n$ submod $\rightsquigarrow$ To Build: $\{e_1, \ldots, e_n\}$ basis for M & $a_1, \ldots, a_r \in R$ with $\{a_1 e_1, \ldots, a_r e_r\}$ ——— N $a_i | a_{i+1} \; \forall i$

Equivalence of matrices

Def: Assume R is a commutative ring & $A, B \in \mathrm{Mat}_{m\times n}(R)$. We say A is equivalent to B ($A \sim B$) if $\exists P \in GL_n(R)$ & $Q \in GL_m(R)$ with $\boxed{B = QAP^{-1}}$

- Clear: $\sim$ defines an equivalence relation on $\mathrm{Mat}_{m\times n}(R)$
- Q: Can we find nice representatives for each class? A: Depends on R

Example: $R = \mathbb{K}$ field

Theorem: Assume R is a PID, then every matrix $A \in \mathrm{Mat}_{m\times n}(R)$ is equivalent to a matrix

$$S = \left(\begin{array}{ccc|c} d_1 & & 0 & \\ & d_2 & & 0 \\ 0 & & \ddots \; d_r & \\ \hline & 0 & & 0 \end{array}\right) \quad \text{with } d_1 \mid d_2 \mid \cdots \mid d_r.$$

(invariant factors)

Name: S = Smith Normal Form of A.