Lecture 31: Rational normal forms, Jordan canonical forms

Last 2 lectures: Saw Classification theorems for fg Torsion modules over PIDs

Fix: $M \neq (0)$ a fg Torsion module over a PID R

Classification Theorem 1: $M = \bigoplus_{p_i \text{ prime}} M_{p_i^{n_i}}$ for suitable $n_i \in \mathbb{Z}_{>0}$ (n_i minimal) with $M_{p_i^{n_i}} \neq \{0\}$.

Furthermore: $M_{p_i^{n_i}} \cong R/(p_i^{v_1^{(i)}}) \oplus \cdots \oplus R/(p_i^{v_{s_i}^{(i)}})$

with $n_i = v_1^{(i)} \geq v_2 \geq \cdots \geq v_s \geq 1$ & the sequence (v_i) is uniquely determined by M & p_i.

Classification Thm v2: $M \simeq R/(q_1) \oplus \cdots \oplus R/(q_r)$ where $q_i \neq 0$, $q_i \notin R^\times$ $\forall i$.

& $q_1 | q_2 | \cdots | q_r$. Furthermore, the sequence of ideals $(q_1), \ldots, (q_r)$ is unique.

TODAY'S GOAL: Focus on the case of $\mathbb{K}[x]$-modules, where $\mathbb{K}$ is a field of characteristic 0 (in char p, perfect fields will be needed (see Math 6112))

Main results: Rational normal form decomposition for $\mathrm{Mat}_{n\times n}(\mathbb{K})$

Jordan canonical form for $\mathrm{Mat}_{n\times n}(\overline{\mathbb{K}})$ eg $\overline{\mathbb{K}} = \mathbb{C}$.

$\boxed{\mathbb{K}[x]\text{-modules}}$

Q: What is a $\mathbb{K}[x]$-module?

Conclude: $\mathbb{K}[x]$-module $\longleftrightarrow$ a $\mathbb{K}$-vector space V + $\varphi \in \mathrm{End}_{\mathbb{K}}(V)$.

From now on, we assume V has $\dim_{\mathbb{K}} V = n < \infty$.

So $\varphi \longleftrightarrow A \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ (matrix of the linear transf. w.r.t a fixed basis)

Def $A, C \in \mathrm{Mat}_{n\times n}(\mathbb{K})$: $A \sim C \iff \exists\, G \in GL_n(\mathbb{K})$ st $A = G^{-1} C G$

$\boxed{\text{The minimal polynomial of } A}$

$\rightsquigarrow$ Define a map $\Psi : \mathbb{K}[X] \longrightarrow \mathbb{K}[A] \subset \mathrm{End}_{\mathbb{K}}(V)$

$$P(X) \longmapsto P(A)$$

<u>Lemma</u>: $\operatorname{Ker} \Psi \neq (0)$ ($\rightsquigarrow$ q_A = monic gen of $\operatorname{Ker} \Psi$ = minimal polynomial)

$\boxed{\text{Cyclic case}}$

Def V is cyclic (as $K[x]$-mod) if $\exists$ $v \in V$ st $\{v, Av, A v, \ldots\}$ spans V.

Proposition Assume $V \neq 0$ is cyclic. Then:

(1) $\deg(q_A)$ is minimal $d > 0$ s.t $\{v, Av, \ldots, A^d v\}$ is ld, ie:

• $\{v, Av, \ldots, A^{d-1}v\}$ is li & • $\{v, Av, \ldots, A^d v\}$ is ld.

(2) Furthermore, in this situation $\{v, Av, \ldots, A^{d-1}v\}$ is a basis for V.

Corollary 1: If V is cyclic and $f_A = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$, then in the basis $B = \{v, Av, \ldots, A^{d-1}v\}$ we have

$$[A]_{BB} = \begin{bmatrix} 0 & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ & & 1 & -a_{d-1} \end{bmatrix}.$$

Corollary 2: If V is cyclic, then: $V \simeq \mathbb{K}[x]/q_A(x)$ (as $\mathbb{K}$-v.s.)

Moreover $q_A(x)$ is independent of the choice of generator v for V

Q: What happens in the non-cyclic case?

Obs: $\dim_{\mathbb{K}} V < \infty$, then V is a torsion module over $\mathbb{K}[x]$.

Rational normal forms

Theorem 1: V $\mathbb{k}$-vector space & $A \in \mathrm{End}_{\mathbb{k}}(V)$ $\quad A \neq 0$. Then, V admits a direct sum decomposition:

$$V = V_1 \oplus \cdots \oplus V_r$$

where each V_i is a cyclic $\mathbb{k}[x]$-module with invariants $q_i \neq 0$, satisfying $q_1 \mid q_2 \mid \cdots \mid q_r$

Furthermore, the sequence $(q_1, \ldots, q_r)$ is uniquely determined by V & A $\quad$ & $q_r = q_A$.

Corollary: V admits a basis $\mathcal{B}$ with

$$[A]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} \boxed{C_{q_1}} & & 0 \\ & \ddots & \\ 0 & & \boxed{C_{q_r}} \end{bmatrix} \qquad C_{q_i} = \text{companion matrix for } q_i$$

This is known as the rational normal form for A. ($A \sim RNF(A)$)

Q: What about alternative Classification Thm?

We factor $q_A(x) = p_1^{n_1}(x) \cdots p_s^{n_s}(x)$ into distinct monic prime powers (Everything is monic, so no unit is needed in the factorization)

Theorem $V = V_{p_1^{n_1}} \oplus \cdots \oplus V_{p_r^{n_r}}$ & each $V_{p_i^{n_i}}$ decomposes further as

$$V_{p_i^{n_i}} \cong \bigoplus_{j=1}^{s_i} \mathbb{K}[x] \big/ (p_i^{\nu_j^{(i)}}) \qquad n_i = \nu_1^{(i)} \geq \cdots \geq \nu_{s_i}^{(i)}$$

(sequence $\nu_j^{(i)}$ is unique)

Jordan canonical form

Assume $\mathbb{K} = \overline{\mathbb{K}}$, char 0 (Eg $\mathbb{K} = \mathbb{C}$) Then $p_i \in \mathbb{K}_{[x]}$ irred & monic $\Rightarrow p_i = (x-\alpha)$

Each $\dfrac{\mathbb{K}[x]}{(p_i)^m}$ piece gives a cyclic submodule $W_{p_i,m} \neq (0)$ of V of dimension m

$W_{p_i,m} = \mathbb{K}[A]\langle w \rangle$

Theorem 3: $W_{p_i,m}$ has a basis B over $\mathbb{K}$ such that

$$[A|_{W_{p_i,m}}]_B = \begin{bmatrix} \alpha_i & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \alpha_i \end{bmatrix} = J(\alpha, m) \qquad (m \times m \text{ matrix})$$

Corollary: Given V & A with $q_A = p_1^{n_1} \cdots p_r^{n_r}$, $\exists$ $\mathcal{B}$ basis for V s.t.

$$[A]_{\mathcal{B}} = \begin{bmatrix} \boxed{A_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{A_r} \end{bmatrix}$$

block diagonal decomp.

Furthermore for $p_i = (x - \alpha_i)$, we have.

$$A_i = \begin{bmatrix} \boxed{J(\alpha_i, m_1^{(i)})} & & 0 \\ & \ddots & \\ 0 & & \boxed{J(\alpha_i, m_{s_i}^{(i)})} \end{bmatrix}$$

with $n_i = m_1^{(i)} \geq \cdots \geq m_{s_i}^{(i)}$

- This block decomposition is the Jordan canonical form of A