# Lecture 32: General Jordan canonical forms - Cayley-Hamilton

**Recall:** Last time we talked about minimal polynomials of matrices

subring gen by $\mathbb{K}$ & $A$. (commutative!)

① Given $A \in Mat_{n\times n}(\mathbb{K})$ $\rightsquigarrow$ $\Psi: \mathbb{K}[x] \twoheadrightarrow \mathbb{K}[A] \subset End_{\mathbb{K}}(\mathbb{K}^n)$

$$P(x) = \sum_{i=0}^{l} a_i x^i \longmapsto P(A) = \sum a_i A^i \qquad (A^0 = Id.)$$

- $\Psi$ ring homomorphism, & $\mathbb{K}$-linear
- $\operatorname{Ker}\Psi \neq (0) \rightsquigarrow \operatorname{Ker}\Psi = (q_A)$ $\quad q_A$ monic in $\mathbb{K}[x]$ = minimal poly of $A$.

**Obs:** $q_A = q_{G^{-1}AG}$ for all $G \in Gl_n(\mathbb{K})$

② We defined Rational Normal forms & Jordan canonical forms (for $\mathbb{K} = \overline{\mathbb{K}}$) viewing $\mathbb{K}^n$ as a $\mathbb{K}[A]$-module ($\rightsquigarrow \mathbb{K}[x]$-module : $(\mathbb{K}^n)_{tor} = \mathbb{K}^n$)

- CASE 1: $\mathbb{K}^n$ cyclic ($\exists\, v \in \mathbb{K}^n$ st $\{v, Av, \ldots, A^k v, \ldots\}$ spans $\mathbb{K}^n$)

- CASE 2: Use classification of modules over PID to reduce to the cyclic case.

## CASE I: $\mathbb{K}^n$ is cyclic  (gen by $\{v, Av, \ldots\}$)

- $q_A = x^n + a_{n-1} x^{n-1} + \cdots + a_0$  $(\deg q_A = \underline{\min}\ d$ st $\{v, Av, \ldots, A^d\}$ is l d$)$

① $\exists\ B = \{v, Av, A^2 v, \ldots, A^{n-1} v\}$ basis with

$$[A]_{BB} = \begin{bmatrix} 0 & & & -a_0 \\ 1 & \ddots & 0 & -a_1 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & -a_{n-1} \end{bmatrix} = C_{q_A}$$

$\shortparallel$

$C_{(E,B)}\, A\, C_{(B,E)}$

(Companion matrix for the polynomial $q_A$.)

$q_A = x^n + a_{n-1} x^{n-1} + \cdots + a_0$

cyclic case!

② If $\mathbb{K} = \overline{\mathbb{K}}$, char $\mathbb{K} = 0$ and $q_A = (x-\alpha)^n$, then $\exists\ w \in \mathbb{K}^n$ st

$$B = \{w, (A-\alpha I)(w), \ldots, (A-\alpha I)^{n-1}(w)\}$$

is a basis for some $v \in \mathbb{K}^n$

$$[A]_{BB} = \begin{bmatrix} \alpha & & & \\ 1 & \alpha & & 0 \\ & 1 & \ddots & \\ 0 & & 1 & \alpha \end{bmatrix} = J_{(\alpha,n)} \quad \text{(Jordan block)}$$

$\boxed{\underline{\text{CASE 2}}: \ \mathbb{K}^n \text{ is not cyclic}}$

We break $\mathbb{K}^n$ into cyclic $\mathbb{K}[A]$-modules

① $\mathbb{K}^n = V_1 \oplus \cdots \oplus V_r$ $\qquad \mathbb{K}[A] \circlearrowright V_i = (\mathbb{K}^n)_{q_i}$ cyclic

$\exists\ B = B_1 \cup \cdots \cup B_r$ $\quad$ ($B_i$ basis for $V_i$) with

$$[A]_{BB} = \begin{bmatrix} C_{q_1} & & 0 \\ & \ddots & \\ 0 & & C_{q_r} \end{bmatrix}$$

with $q_1 \mid q_2 \mid \cdots \mid q_r$ so $q_r = q_A$.

(Rational Normal Form)

② $\mathbb{K} = \overline{\mathbb{K}}$ $\quad$ char $\mathbb{K} = 0$ $\qquad q_A = \prod_{i=1}^{s} (X - \alpha_i)^{m_i}$ $\quad \alpha_i$'s distinct.

$\rightsquigarrow$ $V = V_1' \oplus \cdots \oplus V_s'$ with $V_i' = (\mathbb{K}^n)_{(X-\alpha_i)^{m_i}}$ so $q_{A|_{V_i'}} = (X - \alpha_i)^{m_i}$

$\exists\ B' = B_1' \cup \cdots \cup B_s'$ $\quad$ ($B_i'$ basis for $V_i'$) with

$$[A]_{B'B'} = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{pmatrix} \quad \& \quad A_i = [A|_{V_i'}]_{B_i' B_i'} = \begin{bmatrix} J(\alpha_i, m_1^{(i)}) & & 0 \\ & \ddots & \\ 0 & & J(\alpha_i, m_{s_i}^{(i)}) \end{bmatrix}$$

with $m_i = m_1^{(i)} \geq \cdots \geq m_{s_i}^{(i)}$ $\quad$ (Jordan canonical form)

**Obs:** Written additively, we get : $[A]_{BB} = D + N$

- $D$ = diagonal part. $\rightsquigarrow q_D = \prod_{i=1}^{r} (x-\alpha_i)$
- $N$ = nilpotent $(N^{\dim V} = 0)$. $\rightsquigarrow q_N = x^{\ell}$ for some $\ell$.
- $[A]_{BB}$, $D$ & $N$ commute

$\rightsquigarrow$ more general version of Jordan canonical forms, over perfect fields (Eg char 0).

**Theorem:** Let $\mathbb{K}$ be a perfect field, $n \in \mathbb{Z}_{\geq 1}$, $A \in Mat_{n\times n}(\mathbb{K})$

Then, $\exists!$ $A_S, A_N \in Mat_{n\times n}(\mathbb{K})$ st

(1) $A = A_S + A_N$. (Jordan-Chevalley decomposition of $A$)

(2) $A_S$ & $A_N$ are polynomial in $A$. , ($2^*$) $A, A_S, A_N$ commute

(3) $A_S$ is semisimple & $A_N$ is nilpotent.

- $A_N$ nilpotent means $q_{A_N}(x) = x^{\ell}$ for some $\ell$
- $A_S$ semisimple means $(q_{A_S}, q_{A_S}') = 1$. ($q_A'$ = formal derivative of $q_A$)

(Alternatively $q_{A_S} = \prod_{i=1}^{r} f_i(x)$ $f_i$ = distinct monic irreducibles.)

Q: How to build $A_S$ & $A_N$? $A = A_S + A_N$

$A_N$ nilp, $A_S$ semis

$A_N = P(A)$, $A_S = Q(A)$.

# Characteristic Polynomial

Fix $V$ an $n$-dim'l $\mathbb{K}$-vector space & $A: V \longrightarrow V$ a $k$-linear map.

We have
$$\begin{aligned} K[X] &\longrightarrow K[A] \\ x &\longmapsto A \end{aligned}$$

<u>Def</u>: We define the <u>characteristic polynomial</u> of $A$ as

$$\boxed{\chi_A} = \det(x I_n - A)$$

<u>Obs</u>: If $A \sim C$ (similar, ie $C = G^{-1} A G$ for $G \in GL_n(\mathbb{K})$) then, $\chi_C = \chi_A$.

**Theorem** (Cayley-Hamilton) $\chi_A(A) = 0$ (ie $q_A \mid \chi_A$)

**Lemma:** $\chi_{C_f} = f$ for any monic polynomial $f \in \mathbb{K}[x]$.

Pf/

## • Alternative Proof of CH:

To show: $\underbrace{\chi_A(A)}_{\in \mathrm{End}_{\mathbb{K}}(\mathbb{K}^n)}(v) = 0 \quad \forall v$ in $V$. Assume $\dim V = n$.