Lecture 34: Duals, Tensor products

Recall: Last time we reviewed basics on $\mathbb{K}$-vector spaces.

- Bases = generating sets + l.i ; maximal l.i subsets

Theorem: Let V be a vector space over a field $\mathbb{K}$ with $V \neq \{0\}$.

① Let S be a l.i subset of V. Then there exists a basis B for V with $S \subset B$

② Let Γ be a generating set for V (ie a spanning set). Then, there exists a basis B of V with $B \subset \Gamma$.

Def The dual of V, denoted by V^*, is defined as: $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \underset{\text{1-dim'l vector space.}}{\mathbb{K}})$

Theorem 4: If V is finite-dimensional, then $\dim V^* = \dim V$.

Pf/ Let $\{v_i\}_{1\leq i\leq m}$ be a basis for V. Define $v_i^* \in V^*$ by

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \qquad \left(v_i^*\left(\sum_{j=1}^m a_j v_j\right) = a_i \in \mathbb{K} \quad \text{is } \mathbb{K}\text{-linear} \checkmark\right)$$

Claim: $B^* = \{v_i^*\}_{1\leq i\leq m}$ is a basis for V^* (dual basis)

Remark : $W \subset V$ subspace, then V/W is a vector space. Furthermore:

$$(V/W)^* \longleftrightarrow \{ \xi \in V^* \mid \xi|_W = 0 \}$$

Proposition: $V \xrightarrow{\varphi} (V^*)^*$ (alt notation for $\varphi(v)(f) = \langle v, f \rangle$)

$$v \longmapsto \varphi(v) : V^* \longrightarrow \mathbb{K}$$
$$f \longmapsto f(v)$$

① φ is $\mathbb{K}$-linear & injective

② φ is surjective $\Longleftrightarrow$ $\dim V < \infty$.

$V \xrightarrow{\varphi} (V^*)^*$ $\qquad \varphi(v)(f) = f(v) \quad \forall f \in V^*$

$F' = \{ \xi \in (V^*)^* : \xi|_F = 0 \} \neq \{0\}$

Q: How to dualize a map?

Lemma: If $f: V \longrightarrow W$ is a linear map, then $f^*: W^* \longrightarrow V^*$ defined by $f^*(\xi) = \xi \circ f : V \longrightarrow W \longrightarrow \mathbb{K}$ $\forall \xi \in W^*$ is $\mathbb{K}$-linear

More precisely: $f^*(\xi)(v) = \xi(f(v))$ $\forall v \in V$, $\forall \xi \in W^*$

Prop: If $\dim V = n$, $\dim W = m$ then $[f]_{B_{W^*} B_{V^*}} = [f]^T_{B_V B_W}$.

(HW 12)

Bilinear Maps and Tensor Products

Let V_1, V_2, W be 3 vector spaces over $\mathbb{K}$

<u>Def</u>: A <u>bilinear map</u> $f: V_1 \times V_2 \longrightarrow W$ is a set map which is linear in each coordinate, ie:

- $\forall\, v_1 \in V_1$: $\begin{array}{rcl} V_2 & \longrightarrow & W \\ w & \longmapsto & f(v_1, w) \end{array}$ & $\forall\, v_2 \in V_2$: $\begin{array}{rcl} V_1 & \longrightarrow & W \\ w & \longmapsto & f(w, v_2) \end{array}$

<u>Def</u>: The <u>Tensor product</u> $V_1 \otimes_{\mathbb{K}} V_2$ is a vector space together with a bilinear map

$$\begin{array}{rcl} V_1 \times V_2 & \xrightarrow{\varphi} & V_1 \otimes_{\mathbb{K}} V_2 \\ (v_1, v_2) & \longmapsto & v_1 \otimes v_2 \end{array}$$

satisfying the following <u>universal property</u>: $\forall$ v sp W with $f: V_1 \times V_2 \longrightarrow W$ bilinear we have:

$$\begin{array}{ccc} V_1 \times V_2 & \xrightarrow{f} & W \\ \varphi \big\downarrow & & \\ V_1 \otimes_{\mathbb{K}} V_2 & & \end{array}$$

Construction 1: $\varphi : V_1 \times V_2 \longrightarrow V_1 \otimes_{\mathbb{K}} V_2$ bilinear & $V_1 \otimes_{\mathbb{K}} V_2$ v.sp

- Define $\varphi(v_1, v_2) = v_1 \otimes v_2$ & check it is bilinear