

Lecture 34: Duals, Tensor products

Recall: Last time we reviewed basics in $\mathbb{K}$ -vector spaces.

• Bases = generating sets + l.i. ; maximal l.i. subsets

Theorem: Let V be a vector space over a field $\mathbb{K}$ with $V \neq \{0\}$.

① Let S be a l.i. subset of V . Then there exists a basis B for V with $S \subset B$

② Let Γ be a generating set for V (i.e. a spanning set). Then, there exists a basis B of V with $B \subset \Gamma$.

Def The dual of V , denoted by V^* , is defined as: $V^* = \text{Hom}_{\mathbb{K}}(V, \mathbb{K})$
1-dim'l vector space.

Theorem 4: If V is finite-dimensional, then $\dim V^* = \dim V$.

Prf/ Let $\{v_i\}_{1 \leq i \leq m}$ be a basis for V . Define $v_i^* \in V^*$ by

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \quad (v_i^*(\sum_{j=1}^m a_j v_j) = a_i \in \mathbb{K} \text{ is } \mathbb{K}\text{-linear} \checkmark)$$

Claim: $B^* = \{v_i^*\}_{1 \leq i \leq m}$ is a basis for V^* (dual basis)

Remark : $W \subset V$ subspace, then V/W is a vector space. Furthermore:

$$(V/W)^* \Leftrightarrow \{ \xi \in V^* \mid \xi|_W = 0 \}$$

Proposition: $V \xrightarrow{\varphi} (V^*)^*$ (alt notation for $\varphi(v)(f) = \langle v, f \rangle$)
 $v \longmapsto \varphi(v) : V^* \longrightarrow \mathbb{K}$
 $f \longmapsto f(v)$

① φ is $\mathbb{K}$ -linear & injective

② φ is surjective $\Leftrightarrow \dim V < \infty$.

$$V \xrightarrow{\varphi} (V^*)^*$$

$$\varphi(v)(f) = f(v) \quad \forall f \in V^*$$

$$F' = \{ \xi \in (V^*)^* : \xi|_F = 0 \} \neq \{0\}$$

Q: How to dualize a map?

Lemma: If $f: V \rightarrow W$ is a linear map, then $f^*: W^* \rightarrow V^*$ defined by $f^*(\xi) = \xi \circ f: V \rightarrow W \rightarrow \mathbb{K} \quad \forall \xi \in W^*$ is $\mathbb{K}$ -linear

More precisely: $f^*(\xi)(v) = \xi(f(v)) \quad \forall v \in V, \forall \xi \in W^*$

Prop: If $\dim V = n$, $\dim W = m$ then $[f]_{B_{W^*} B_{V^*}} = [f]_{B_V B_W}^T$.
(HW 12)

Bilinear Maps and Tensor Products

Let V_1, V_2, W be 3 vector spaces over $\mathbb{K}$

Def: A bilinear map $f: V_1 \times V_2 \longrightarrow W$ is a set map which is linear in each coordinate, i.e.:

$$\bullet \forall v_1 \in V_1: V_2 \longrightarrow W \quad \& \quad \bullet \forall v_2 \in V_2: V_1 \longrightarrow W$$

$$w \longmapsto f(v_1, w) \qquad \qquad \qquad w \longmapsto f(w, v_2)$$

Def: The tensor product $V_1 \otimes_{\mathbb{K}} V_2$ is a vector space together

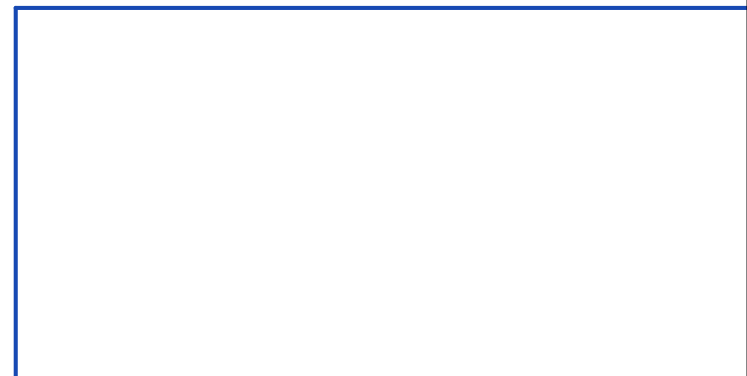
with a bilinear map

$$V_1 \times V_2 \xrightarrow{\varphi} V_1 \otimes_{\mathbb{K}} V_2$$

$$(v_1, v_2) \longmapsto v_1 \otimes v_2$$

satisfying the following universal property: $\forall$ v-sp W with $f: V_1 \times V_2 \longrightarrow W$ bilinear we have:

$$\begin{array}{ccc} V_1 \times V_2 & \xrightarrow{f} & W \\ \varphi \downarrow & & \\ V_1 \otimes_{\mathbb{K}} V_2 & & \end{array}$$



Instruction 1: $\varphi: V_1 \times V_2 \longrightarrow V_1 \underset{\mathbb{K}}{\otimes} V_2$ bilinear & $V_1 \underset{\mathbb{K}}{\otimes} V_2$ v.sp

• Define $\varphi(v_1, v_2) = v_1 \otimes v_2$ & check it is bilinear