Lecture 34: Duals, Tensor products

Recall: Last time we reviewed basics on $\mathbb{K}$-vector spaces.

- Bases = generating sets + l.i ; maximal l.i subsets

Theorem: Let V be a vector space over a field $\mathbb{K}$ with $V \neq \{0\}$.

① Let S be a l.i subset of V. Then there exists a basis B for V with $S \subset B$

② Let Γ be a generating set for V (ie a spanning set). Then, there exists a basis B of V with $B \subset \Gamma$.

Def The dual of V, denoted by V^*, is defined as: $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$
1-dim'l vector space.

Theorem 4: If V is finite-dimensional, then $\dim V^* = \dim V$.

Pf/ Let $\{v_i\}_{1\leq i\leq m}$ be a basis for V. Define $v_i^* \in V^*$ by

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \qquad \left(v_i^*\left(\sum_{j=1}^m a_j v_j\right) = a_i \in \mathbb{K} \quad \text{is } \mathbb{K}\text{-linear} \checkmark\right)$$

Claim: $B^* = \{v_i^*\}_{1\leq i\leq m}$ is a basis for V^* (dual basis)

Remark: $W \subset V$ subspace, then V/W is a vector space. Furthermore:

$$(V/W)^* \longleftrightarrow \{ \xi \in V^* \mid \xi|_W = 0 \}$$

Proposition:

$$\begin{aligned} V &\xrightarrow{\varphi} (V^*)^* \\ v &\longmapsto \varphi(v) : V^* \longrightarrow \mathbb{K} \\ &\qquad\qquad\quad f \longmapsto f(v) \end{aligned}$$

(alt notation for $\varphi(v)(f) = \langle v, f \rangle$)

① φ is $\mathbb{K}$-linear & injective

② φ is surjective $\Longleftrightarrow$ $\dim V < \infty$.

Proof: . $\varphi(v) \in (V^*)^*$ is clear

. φ is linear:

- $\varphi(v+v')(f) = f(v+v') = f(v) + f(v') = (\varphi(v) + \varphi(v'))(f)$

This is true for all f, so $\varphi(v+v') = \varphi(v) + \varphi(v')$ in $(V^*)^*$.

- $\varphi(zv)(f) = f(zv) = z f(v) = z(\varphi(v)(f)) = (z\varphi(v))(f)$

This is true for all f, so $\varphi(zv) = z \cdot \varphi(z)$ in $(V^*)^*$.

$$V \xrightarrow{\varphi} (V^*)^* \qquad \varphi(v)(f) = f(v) \quad \forall f \in V^*$$

- φ is injective $\quad (\operatorname{Ker}\varphi = \{0\})$

$\varphi(v) = 0$ means $\varphi(v)(f) = 0 \quad \forall f$.

If $v \neq 0$, then by Theorem 3 we can find a basis B for V with $v \in V$.

$\Rightarrow \exists\, v^* : V \longrightarrow 0$ with $v^*(w) = \begin{cases} 0 & w \in B \setminus \{v\} \\ 1 & w = v \end{cases}$ for $w \in B$.

Then $\varphi(v)(v^*) = v^*(v) = 1 \neq 0$ Contr!

Conclude: $v = 0$, so $\operatorname{Ker}\varphi = \{0\}$.

- We saw $V \simeq V^*$ if $\dim V < \infty$, so V & $(V^*)^*$ have $=$ dim. $\Rightarrow \varphi$ surj.
- Converse: if $\dim V$ is infinite V with basis $B = \{v_i : i \in I\}$.

$$V^* = \left\{ \sum_{i \in I} a_i v_i^* \right\} \underset{\neq}{\supset} F = \left\{ \sum_{i \in I} a_i v_i^* : a_i = 0 \text{ for all but finitely many } i \right\}$$

Claim: Any linear form $\xi : V^* \longrightarrow K$ which vanishes on F cannot be in $\operatorname{Im}\varphi$.

Pf/ $F' = \{ \xi \in (V^*)^* : \xi|_F = 0 \} \neq \{0\}$

Take $\{v_i^*\}$ & complete to a basis B' for V^*. Then $w^* \in F'$ for any $w \in B' \setminus \{v_i^*\}_{i \in I}$

$F' = \{ \xi \in (V^*)^* : \xi|_F = 0 \} \neq \{0\}$

Set $H = \{ v \in V : \varphi(v)|_F = 0 \ \forall \varphi \in F \}$

Then, $F' \cap \operatorname{Im} \varphi \subset \varphi(H) = 0$ so $\emptyset \neq F' \setminus \{0\} \subset (V^*)^* \setminus \operatorname{Im} \varphi$ $\square$

(easy calculation). [see Notes]

Q: How to dualize a map?

Lemma: If $f: V \longrightarrow W$ is a linear map, then $f^*: W^* \longrightarrow V^*$ defined by $f^*(\xi) = \xi \circ f : V \longrightarrow W \longrightarrow \mathbb{K}$ $\forall \xi \in W^*$ is $\mathbb{K}$-linear

More precisely: $f^*(\xi)(v) = \xi(f(v))$ $\forall v \in V, \forall \xi \in W^*$

Prop: If $\dim V = n$, $\dim W = m$ then $[f^*]_{(B_W)^*(B_V)^*} = [f]^T_{B_V B_W}$.
(HW 12)

Bilinear Maps and Tensor Products

Let V_1, V_2, W be 3 vector spaces over $\mathbb{K}$

Def: A <u>bilinear map</u> $f: V_1 \times V_2 \longrightarrow W$ is a set map which is linear in each coordinate, ie:

- $\forall\, v_1 \in V_1$: $\begin{array}{rcl} V_2 & \longrightarrow & W \\ w & \longmapsto & f(v_1, w) \end{array}$ $\in \mathrm{Hom}_{\mathbb{K}}(V_2, W)$

&

- $\forall\, v_2 \in V_2$: $\begin{array}{rcl} V_1 & \longrightarrow & W \\ w & \longmapsto & f(w, v_2) \end{array}$ $\in \mathrm{Hom}_{\mathbb{K}}(V_1, W)$

Def: The <u>Tensor product</u> $V_1 \otimes_{\mathbb{K}} V_2$ is a vector space together with a bilinear map

$$\begin{array}{rcl} V_1 \times V_2 & \xrightarrow{\varphi} & V_1 \otimes_{\mathbb{K}} V_2 \\ (v_1, v_2) & \longmapsto & v_1 \otimes v_2 \end{array}$$

"indecomposable tensor"

satisfying the following <u>universal property</u>: $\forall$ v sp W with $f: V_1 \times V_2 \longrightarrow W$ bilinear we have:

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "f"', "\text{bilinear}"] \arrow[d, "\text{bilinear}\ \varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & W \\ V_1 \otimes_{\mathbb{K}} V_2 \arrow[ur, dashed, "\exists!\, \tilde{f}\ \text{linear}"'] & \end{tikzcd}$$

Ex: $V \otimes_{\mathbb{K}} \mathbb{K} = V = \mathbb{K} \otimes_{\mathbb{K}} V$

$v \otimes 1 \longleftrightarrow v \longleftrightarrow 1 \otimes v$

$\tilde{f}_{(v)} = f_{(v,1)}$ resp $f(1, v)$.

$\varphi(v, a) = a \cdot v$.

Construction 1: $\varphi: V_1 \times V_2 \longrightarrow V_1 \otimes_{\mathbb{K}} V_2$ bilinear & $V_1 \otimes_{\mathbb{K}} V_2$ v.sp

- $V_1 \otimes_{\mathbb{K}} V_2$ is a vector space spanned by $\{(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\}$ quotiented by the subspace H spanned by:

$$\text{①} \quad (z_1 v_1 + z_1' v_1', v_2) - z_1 (v_1, v_2) - z_1' (v_1', v_2)$$

&

$$\text{②} \quad (v_1, z_2 v_2 + z_2' v_2') - z_2 (v_1, v_2) - z_2' (v_1, v_2')$$

So in the quotient:
$$(z_1 v_1 + z_1' v_1') \otimes v_2 = z_1 (v_1 \otimes v_2) + z_1' (v_1' \otimes v_2).$$
$$v_1 \otimes (z_2 v_2 + z_2' v_2') = z_2 (v_1 \otimes v_2) + z_2' (v_1 \otimes v_2')$$

- Define $\varphi(v_1, v_2) = v_1 \otimes v_2$ & check it is bilinear

①' $\varphi(z_1 v_1 + z_1' v_1', v_2) \overset{?}{=} z_1 \varphi(v_1, v_2) + z_1' \varphi(v_1', v_2)$ this is ①!

②' $\varphi(v_1, z_2 v_2 + z_2' v_2') \overset{?}{=} z_2 \varphi(v_1, v_2) + z_2' \varphi(v_1, v_2')$ —— ②!

⚠ The map is bilinear because we defined $V_1 \otimes_{\mathbb{K}} V_2$ as a quotient of vector spaces.