Lecture 35: Tensor products, Hom-Tensor adjointness

Recall: Last time we define tensor products & duals of vector spaces & maps

Def The dual of V, denoted by V^*, is defined as: $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \underset{\text{1-dim'l vector space.}}{\mathbb{K}})$

Theorem: If V is finite-dimensional, then $\dim V^* = \dim V$. $\left(\underset{\text{basis}}{\mathcal{B}} \rightsquigarrow \underset{\text{basis}}{\mathcal{B}^* \text{ dual}}\right)$

Q: How to dualize a map?

Lemma: If $f: V \longrightarrow W$ is a linear map, then $f^*: W^* \longrightarrow V^*$ defined by $f^*(\xi) = \xi \circ f : V \longrightarrow W \longrightarrow \mathbb{K}$ $\forall \xi \in W^*$ is $\mathbb{K}$-linear

More precisely: $f^*(\xi)(v) = \xi(f(v))$ $\forall v \in V, \forall \xi \in W^*$

Prop (HW 12): If $\dim V = n$, $\dim W = m$ then $[f]_{(\mathcal{B}_W)^*(\mathcal{B}_V)^*} = [f]^T_{\mathcal{B}_V \mathcal{B}_W}$.

Theorem 2 $V \xrightarrow{\varphi} (V^*)^*$ linear; surj $\Leftrightarrow \dim V < \infty$

$v \longmapsto \varphi(v) : (f \longmapsto f(v))$ $\forall f \in V^*$;

Bilinear Maps and Tensor Products

Let V_1, V_2, W be 3 vector spaces over $\mathbb{K}$

<u>Def</u>: A <u>bilinear map</u> $f: V_1 \times V_2 \longrightarrow W$ is a set map which is linear in each coordinate, ie:

- $\forall v_1 \in V_1$: $\begin{aligned} V_2 &\longrightarrow W \\ w &\longmapsto f(v_1, w) \end{aligned}$ $\in \mathrm{Hom}_{\mathbb{K}}(V_2, W)$ & $\forall v_2 \in V_2$: $\begin{aligned} V_1 &\longrightarrow W \\ w &\longmapsto f(w, v_2) \end{aligned}$ $\in \mathrm{Hom}_{\mathbb{K}}(V_1, W)$

<u>Def</u>: The <u>Tensor product</u> $V_1 \otimes_{\mathbb{K}} V_2$ is a vector space together with a bilinear map

$$\begin{aligned} V_1 \times V_2 &\xrightarrow{\varphi} V_1 \otimes_{\mathbb{K}} V_2 \\ (v_1, v_2) &\longmapsto v_1 \otimes v_2 \end{aligned}$$

"indecomposable tensor"

satisfying the following <u>universal property</u>: $\forall$ v sp W with $f: V_1 \times V_2 \longrightarrow W$ bilinear we have:

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "f"', "\text{bilinear}"] \arrow[d, "\varphi", "\text{bilinear}"'] \arrow[dr, phantom, "\circlearrowleft"] & W \\ V_1 \otimes_{\mathbb{K}} V_2 \arrow[ur, dashed, "\exists! \tilde{f} \text{ linear}"'] & \end{tikzcd}$$

<u>Ex.</u>: $V \otimes_{\mathbb{K}} \mathbb{K} = V = \mathbb{K} \otimes_{\mathbb{K}} V$

$v \otimes 1 \leftrightarrow v \leftrightarrow 1 \otimes v$

$\tilde{f}_{(v)} = f_{(v,1)}$ resp $f(1, v)$.

$\varphi(v, a) = a \cdot v$.

Construction!: $\varphi : V_1 \times V_2 \longrightarrow V_1 \underset{\mathbb{K}}{\otimes} V_2$ bilinear & $V_1 \underset{\mathbb{K}}{\otimes} V_2$ v.sp

- $V_1 \underset{\mathbb{K}}{\otimes} V_2 = \left(\bigoplus_{\substack{v_1 \in V_1 \\ v_2 \in V_2}} \mathbb{K}(v_1, v_2) \right) \Big/ H$ (quotient of v.sp, so v.sp)

H = Subspace spanned by:

& ① $(z_1 v_1 + z_1' v_1', v_2) - z_1(v_1, v_2) - z_1'(v_1', v_2)$

② $(v_1, z_2 v_2 + z_2' v_2') - z_2(v_1, v_2) - z_2'(v_1, v_2')$

So in the quotient: $(z_1 v_1 + z_1' v_1') \otimes v_2 = z_1(v_1 \otimes v_2) + z_1'(v_1' \otimes v_2)$.

$(v_1 \otimes (z_2 v_2 + z_2' v_2') = z_2(v_1 \otimes v_2) + z_2'(v_1 \otimes v_2')$

- Define $\varphi : V_1 \times V_2 \longrightarrow V_1 \underset{\mathbb{K}}{\otimes} V_2$ as $\varphi(v_1, v_2) = \overline{(v_1, v_2)} =: v_1 \otimes v_2$

①' $\varphi(z_1 v_1 + z_1' v_1', v_2) \overset{?}{=} z_1 \varphi(v_1, v_2) + z_1' \varphi(v_1', v_2)$ this is ①!

②' $\varphi(v_1, z_2 v_2 + z_2' v_2') \overset{?}{=} z_2 \varphi(v_1, v_2) + z_2' \varphi(v_1, v_2')$ — ②!

bilinear

⚠ The map is bilinear because we defined $V_1 \otimes_{\mathbb{K}} V_2$ as a quotient of vector spaces.

Construction II:

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "f"] \arrow[d, "\varphi"'] & W \\ V_1 \otimes_{\mathbb{K}} V_2 \arrow[ur, dashed, "\exists ! \tilde{f} \text{ linear}"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

f bilinear ✓

φ bilinear ✓

$$V_1 \otimes_{\mathbb{K}} V_2 = \frac{\bigoplus_{v_i \in V_i} \mathbb{K}(v_1, v_2)}{H}$$

<u>Need</u>: $\tilde{f}(v_1 \otimes v_2) = \tilde{f}(\varphi(v_1, v_2)) = f(v_1, v_2)$

Proposition: If $f_1 : V_1 \longrightarrow W_1$ & $f_2 : V_2 \longrightarrow W_2$ are $\mathbb{K}$-linear, then

$$\begin{tikzcd} (f_1, f_2) : V_1 \times V_2 \arrow[r] & W_1 \times W_2 \arrow[d] \\ & W_1 \otimes W_2 \end{tikzcd}$$

Lemma: $\dim_{\mathbb{K}}(V_1 \otimes V_2) = \dim_{\mathbb{K}} V_1 \cdot \dim_{\mathbb{K}} V_2$ (product of cardinalities)

$$B = \left\{ v_i^{(1)} \otimes v_j^{(2)} \quad \begin{matrix} i \in I_1 \\ j \in I_2 \end{matrix} \right\} \text{ is a basis for } V_1 \otimes V_2$$

Remark: Assume $\dim V_i = n_i < \infty$ & $\dim W_i = m_i < \infty$.

Assume $f_1 : V_1 \longrightarrow W_1$ linear are identified with matrices $X_1 \in \mathrm{Mat}_{m_1 \times n_1}(\mathbb{K})$

$f_2 : V_2 \longrightarrow W_2$ — & $X_2 \in \mathrm{Mat}_{m_2 \times n_2}(\mathbb{K})$

Then: $f_1 \otimes f_2$ gets identified with a matrix $X_1 \otimes X_2 \in \mathrm{Mat}_{(m_1+m_2)\times(n_1+n_2)}(\mathbb{K})$.

More precisely, if $X_1 = \begin{bmatrix} a_{11} & \cdots & a_{1n_1} \\ \vdots & & \\ a_{m_1 1} & \cdots & a_{m_1 n_1} \end{bmatrix}$ & $X_2 = \begin{bmatrix} b_{11} & \cdots & b_{1n_2} \\ \vdots & & \\ b_{m_2 1} & \cdots & b_{m_2 n_2} \end{bmatrix}$, then

$B_{V_1} = \{ v_i^{(1)} \ i=1,\dots,n_1 \}$

$B_{V_2} = \{ v_i^{(2)} \ i=1,\dots,n_2 \}$

$B_{W_1} = \{ w_i^{(1)} \ i=1,\dots,m_1 \}$

$B_{W_2} = \{ w_i^{(2)} \ i=1,\dots,m_2 \}$

$$X_1 \otimes X_2 =$$

Here we use $B_{V_1 \otimes V_2} = B_{V_1} \times B_{V_2} =$

$B_{W_1 \otimes W_2} = B_{W_1} \times B_{W_2} =$

Hom-Tensor adjointness:

Prop: There is a natural map $V^* \otimes W \xrightarrow{\Phi} \mathrm{Hom}(V,W)$ with $\Phi(\xi \otimes w) : v \longmapsto \underbrace{\xi(v)}_{\in \mathbb{K}} \cdot w$ & Φ is injective.

Moreover, Φ is an isomorphism if V is finite-dimensional

Have $\Phi\left(\sum_{j=1}^{N} \xi_j \otimes \omega_j\right) = 0$ & $\{\omega_1, \ldots \omega_N\}$ are li.