Lecture 35: Tensor products, Hom-Tensor adjointness

Recall: Last time we define tensor products & duals of vector spaces & maps

Def The dual of V, denoted by V^*, is defined as: $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$

1-dim'l vector space.

Theorem: If V is finite-dimensional, then $\dim V^* = \dim V$. ($\underset{\text{basis}}{\mathcal{B}} \rightsquigarrow \underset{\text{basis}}{\mathcal{B}^*}$ dual)

Q: How to dualize a map?

Lemma: If $f: V \longrightarrow W$ is a linear map, then $f^*: W^* \longrightarrow V^*$
defined by $f^*(\xi) = \xi \circ f : V \longrightarrow W \longrightarrow \mathbb{K}$ $\forall \xi \in W^*$ is $\mathbb{K}$-linear

More precisely: $f^*(\xi)(v) = \xi(f(v))$ $\forall v \in V$, $\forall \xi \in W^*$

Prop: If $\dim V = n$, $\dim W = m$ then $[f]_{(\mathcal{B}_W)^*(\mathcal{B}_V)^*} = [f]^T_{\mathcal{B}_V \mathcal{B}_W}$.
(HW 12)

Theorem 2 $V \xrightarrow{\varphi} (V^*)^*$ linear
$v \longmapsto \varphi(v) : (f \longmapsto f(v))$ $\forall f \in V^*$; surj $\Leftrightarrow \dim V < \infty$

Bilinear Maps and Tensor Products

Let V_1, V_2, W be 3 vector spaces over $\mathbb{K}$

<u>Def</u>: A <u>bilinear map</u> $f: V_1 \times V_2 \longrightarrow W$ is a set map which is linear in each coordinate, ie:

- $\forall v_1 \in V_1$: $\begin{aligned} V_2 &\longrightarrow W \\ w &\longmapsto f(v_1, w) \end{aligned}$ $\in Hom_{\mathbb{K}}(V_2, W)$ & $\forall v_2 \in V_2$: $\begin{aligned} V_1 &\longrightarrow W \\ w &\longmapsto f(w, v_2) \end{aligned}$ $\in Hom_{\mathbb{K}}(V_1, W)$

<u>Def</u>: The <u>Tensor product</u> $V_1 \otimes_{\mathbb{K}} V_2$ is a vector space together with a bilinear map

$$\begin{aligned} V_1 \times V_2 &\xrightarrow{\varphi} V_1 \otimes_{\mathbb{K}} V_2 \\ (v_1, v_2) &\longmapsto v_1 \otimes v_2 \end{aligned}$$

"indecomposable tensor"

satisfying the following <u>universal property</u>: $\forall$ v sp W with $f: V_1 \times V_2 \longrightarrow W$ bilinear we have:

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "f"', "\text{bilinear}"] \arrow[d, "\text{bilinear } \varphi"'] & W \\ V_1 \otimes_{\mathbb{K}} V_2 \arrow[ur, dashed, "\exists! \tilde{f} \text{ linear}"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

<u>Ex</u>: $V \otimes_{\mathbb{K}} \mathbb{K} = V = \mathbb{K} \otimes_{\mathbb{K}} V$

$v \otimes 1 \longleftrightarrow v \longleftrightarrow 1 \otimes v$

$\tilde{f}_{(v)} = f_{(v,1)}$ resp $f(1, v)$.

$\varphi(v, a) = a \cdot v$.

Construction!: $\varphi : V_1 \times V_2 \longrightarrow V_1 \underset{\mathbb{K}}{\otimes} V_2$ bilinear & $V_1 \underset{\mathbb{K}}{\otimes} V_2$ v.sp

- $$V_1 \otimes_{\mathbb{K}} V_2 = \left(\bigoplus_{\substack{v_1 \in V_1 \\ v_2 \in V_2}} \mathbb{K}(v_1, v_2) \right) \Big/ H$$ (quotient of v.sp, so v.sp)

H = Subspace spanned by:

① $(z_1 v_1 + z_1' v_1', v_2) - z_1 (v_1, v_2) - z_1' (v_1', v_2)$

&

② $(v_1, z_2 v_2 + z_2' v_2') - z_2 (v_1, v_2) - z_2' (v_1, v_2')$

So in the quotient: $(z_1 v_1 + z_1' v_1') \otimes v_2 = z_1 (v_1 \otimes v_2) + z_1' (v_1' \otimes v_2)$.

$v_1 \otimes (z_2 v_2 + z_2' v_2') = z_2 (v_1 \otimes v_2) + z_2' (v_1 \otimes v_2')$

- Define $\varphi : V_1 \times V_2 \longrightarrow V_1 \underset{\mathbb{K}}{\otimes} V_2$ as $\varphi(v_1, v_2) = \overline{(v_1, v_2)} =: v_1 \otimes v_2$

①' $\varphi(z_1 v_1 + z_1' v_1', v_2) \overset{?}{=} z_1 \varphi(v_1, v_2) + z_1' \varphi(v_1', v_2)$ this is ①!

②' $\varphi(v_1, z_2 v_2 + z_2' v_2') \overset{?}{=} z_2 \varphi(v_1, v_2) + z_2' \varphi(v_1, v_2')$ — ②!

bilinear

⚠ The map is bilinear because we defined $V_1 \otimes_{\mathbb{K}} V_2$ as a quotient of vector spaces.

Construction II:

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & W \\ V_1 \otimes_{\mathbb{K}} V_2 \arrow[ur, dashed, "\exists! \tilde{f} \text{ linear}"'] & \end{tikzcd}$$

f bilinear ✓

φ bilinear ✓

$$V_1 \otimes_{\mathbb{K}} V_2 = \frac{\bigoplus_{v_i \in V_i} \mathbb{K}(v_1, v_2)}{H}$$

Need: $\tilde{f}(v_1 \otimes v_2) = \tilde{f}(\varphi(v_1, v_2)) = f(v_1, v_2)$

We can use the universal property of $\bigoplus$ to get a ! linear map

$$g : \bigoplus_{v_i \in V_i} \mathbb{K}(v_1, v_2) \longrightarrow W \qquad \text{with} \quad g(v_1, v_2) = f(v_1, v_2)$$

Now we want to check $g|_H = 0$.

(1'') $g((z_1 v_1 + z_1' v_1', v_2)) = f((z_1 v_1 + z_1' v_1', v_2))$

$(f \text{ bil}) = z_1 f(v_1, v_2) + z_1' f(v_1', v_2) = z_1 g(v_1, v_2) + z_1' g(v_1', v_2)$

So $g((z_1 v_1 + z_1' v_2', v_2) - z_1(v_1, v_2) - z_1'(v_1', v_2)) = 0$

(2'') Similarly $g((v_1, z_2 v_2 + z_2' v_2) - z_2(v_1, v_2) - z_2'(v_1, v_2')) = 0$

Conclusion: $H \subseteq \ker g$ so g gives a unique linear map $\tilde{f} : V_1 \otimes_{\mathbb{K}} V_2 \longrightarrow W$ making the diagram commute $[\, \tilde{f}(v_1 \otimes v_2) = g(v_1 \otimes v_2) = f(v_1, v_2) \,]$

Proposition: If $f_1 : V_1 \longrightarrow W_1$ & $f_2 : V_2 \longrightarrow W_2$ are $\mathbb{K}$-linear, then

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "(f_1, f_2)"] \arrow[dr, dashed, "\text{bilinear}"'] & W_1 \times W_2 \arrow[d, "\text{bilinear}"] \\ & W_1 \otimes W_2 \end{tikzcd}$$

$\rightsquigarrow \exists!\ f_1 \otimes_{\mathbb{K}} f_2 : V_1 \otimes_{\mathbb{K}} V_2 \longrightarrow V_1 \otimes_{\mathbb{K}} W_2$ linear.

where $(v_1 \otimes v_2) \longmapsto f_1(v_1) \otimes f_2(v_2)$

Lemma: $\dim_{\mathbb{K}} (V_1 \otimes V_2) = \dim_{\mathbb{K}} V_1 \cdot \dim_{\mathbb{K}} V_2$ (product of cardinalities)

Pf/ If $B_{V_1} = \{ v_i^{(1)} : i \in I_1 \}$ is a basis for V_1 &
$B_{V_2} = \{ v_i^{(2)} : i \in I_2 \}$ ——— V_2, then

Claim $B = \{ v_i^{(1)} \otimes v_j^{(2)} \ \ i \in I_1, j \in I_2 \}$ is a basis for $V_1 \otimes V_2$

- B spans $V_1 \otimes V_2$: Enough to write $v_1 \otimes v_2 \in \mathrm{Sp}(B)$ for $v_1 \in V_1$, $v_2 \in V_2$

$v_1 = \sum_{i \in I_1} a_i v_i^{(1)}$ & $v_2 = \sum_{j \in I_2} b_j v_j^{(2)}$. with $a_i = 0$, $b_j = 0$ $\forall (i,j)$ but fin many.

Then $v_1 \otimes v_2 = \Big(\underbrace{\sum_{i \in I_1}}_{\text{finite}} a_i v_i^{(1)} \Big) \otimes \Big(\underbrace{\sum_{j \in I_2}}_{\text{finite}} b_j v_j^{(2)} \Big) \underset{\text{Reln (1)}}{=} \underbrace{\sum_{i \in I_1}}_{\text{finite}} a_i \Big(v_i^{(1)} \otimes \Big(\sum_{j \in I_2} b_j v_j^{(2)} \Big) \Big)$

$\underset{\text{Reln (2)}}{=} \sum_{\substack{i \in I_1 \\ j \in I_2 \\ \text{finite}}} \underbrace{a_i b_j}_{\in \mathbb{K}} \big(v_i^{(1)} \otimes v_j^{(2)} \big)$ and $a_i b_j = 0$ for all but finitely many $i \in I_1, j \in I_2$

$B = \{ v_i^{(1)} \otimes v_j^{(2)} \quad i \in I_1, j \in I_2 \}$ is a basis for $V_1 \otimes V_2$

- B is l.i.: $\displaystyle\sum_{\substack{i \in I_1, j \in I_2 \\ \text{finite}}} c_{ij}\, v_i^{(1)} \otimes v_j^{(2)} = 0 \in V_1 \otimes V_2$. want to show $c_{ij} = 0$

Use def to rewrite it as $\displaystyle\sum_{\substack{j \in I_2 \\ \text{finite}}} \underbrace{\Big(\sum_{\substack{i \in I_1 \\ \text{finite}}} c_{ij}\, v_i^{(1)} \Big)}_{\in V_1} \otimes \underbrace{v_j^{(2)}}_{\in V_2} = 0.$ (*) $\rightsquigarrow$ Pick $l \in I_2$ & show $c_{il} = 0 \ \forall i$

Since B_{V_2} is a basis for V_2: $\exists\ \Psi : V_2 \to K$ linear with
$\Psi(v_j^{(2)}) = \delta_{j,l}$ $\qquad \Psi = (v_l^{(2)})^* \in V_2^*$.

By Proposition $\exists\ id_{V_1} \otimes \Psi : V_1 \otimes V_2 \to \underbrace{V_1 \otimes_K K}_{= V_1}$ linear with $(id_{V_1} \otimes \Psi)(v_1 \otimes v_2) = v_1 \otimes \Psi(v_2)$.

Apply $id_{V_1} \otimes \Psi$ to (*). Then:

$$\underset{V_1 \otimes_K K = V_1}{\overset{\cap}{0}} = (id_{V_1} \otimes \Psi)(0) = \sum_{\substack{j \in I_2 \\ \text{finite}}} \Big(\sum_{\substack{i \in I_1 \\ \text{finite}}} c_{ij}\, v_i^{(1)} \Big) \otimes \Psi(v_j^{(2)}) \overset{\text{only } j=l \text{ survives.}}{=} \sum_{i \in I_1} c_{il}\, v_i^{(1)} \otimes 1$$

$$\Rightarrow \underset{V_1}{0} = \sum_{\substack{i \in I_1 \\ \text{finite}}} c_{il}\, v_i^{(1)} \underset{B_{V_1} \text{ basis}}{\Longrightarrow} c_{il} = 0 \quad \forall i$$

$\square$

Remark: Assume $\dim V_i = n_i < \infty$ & $\dim W_i = m_i < \infty$.

Assume $f_1 : V_1 \longrightarrow W_1$ linear are identified with matrices $X_1 \in \mathrm{Mat}_{m_1 \times n_1}(\mathbb{K})$

$f_2 : V_2 \longrightarrow W_2$ — & $X_2 \in \mathrm{Mat}_{m_2 \times n_2}(\mathbb{K})$

Then: $f_1 \otimes f_2$ gets identified with a matrix $X_1 \otimes X_2 \in \mathrm{Mat}_{(m_1 \cdot m_2) \times (n_1 \cdot n_2)}(\mathbb{K})$.

More precisely, if $X_1 = \begin{bmatrix} a_{11} & \cdots & a_{1n_1} \\ \vdots & & \\ a_{m_1 1} & \cdots & a_{m_1 n_1} \end{bmatrix}$ & $X_2 = \begin{bmatrix} b_{11} & \cdots & b_{1n_2} \\ \vdots & & \\ b_{m_2 1} & \cdots & b_{m_2 n_2} \end{bmatrix}$, then

$$X_1 \otimes X_2 = \begin{bmatrix} a_{11} X_2 & \cdots & a_{1n_1} X_2 \\ \vdots & & \\ \underbrace{\boxed{a_{m_1 1} X_2}}_{m_2 \times n_2 \text{ matrix}} & \cdots & a_{m_1 n_1} X_2 \end{bmatrix}$$

$B_{V_1} = \{ v_i^{(1)} \ i = 1, \ldots, n_1 \}$

$B_{V_2} = \{ v_i^{(2)} \ i = 1, \ldots, n_2 \}$

$B_{W_1} = \{ w_i^{(1)} \ i = 1, \ldots, m_1 \}$

$B_{W_2} = \{ w_i^{(2)} \ i = 1, \ldots, m_2 \}$

Here we use $B_{V_1 \otimes V_2} = B_{V_1} \times B_{V_2} = \bigcup_{i=1}^{n_1} \{ v_i^{(1)} \} \times \{ v_j^{(2)} : j = 1, \ldots, n_2 \}$

$B_{W_1 \otimes W_2} = B_{W_1} \times B_{W_2} = \bigcup_{i=1}^{m_1} \{ w_i^{(1)} \} \times \{ w_j^{(2)} : j = 1, \ldots, m_2 \}$

Hom-Tensor adjointness

<u>Prop</u>: There is a natural map $V^* \otimes_{\mathbb{K}} W \xrightarrow{\Phi} \mathrm{Hom}_{\mathbb{K}}(V,W)$ with $\Phi(\xi \otimes w) : v \longmapsto \underbrace{\xi(v)}_{\in \mathbb{K}} \cdot w$ & Φ is injective.

Moreover, Φ is an isomorphism if V is finite-dimensional ($\dim V^* \otimes W = \dim V \cdot \dim V = \dim \mathrm{Hom}_{\mathbb{K}}(V,W)$)

<u>Proof</u> Define

$$\begin{aligned} \varphi : V^* \times W &\longrightarrow \mathrm{Hom}_{\mathbb{K}}(V,W) \\ (\xi, w) &\longmapsto \{ v \longmapsto \xi(v) w \} \end{aligned}$$

- <u>Easy check</u>: φ is bilinear. Hence, it yields a unique linear map

$$\phi : V^* \otimes_{\mathbb{K}} W \longrightarrow \mathrm{Hom}_{\mathbb{K}}(V,W)$$

with $\phi(\xi \otimes w) = \varphi(\xi, w)$.

- <u>φ is injective</u>: Let $\alpha \in V^* \otimes W$ with $\phi(\alpha) = 0$. Set $\alpha = \sum_{j=1}^{N} \xi_j \otimes w_j$ (absorb scalars into ξ_j)

We can <u>assume</u> $w_1, \dots, w_N$ are <u>linearly independent</u>

(Otherwise, eg $w_N = a_1 w_1 + \dots + a_{N-1} w_{N-1} \rightsquigarrow \alpha = \sum_{j=1}^{N-1} \xi_j \otimes w_j + \xi_N \otimes \sum_{j=1}^{N-1} a_j w_j$

$= \sum_{j=1}^{N-1} (\xi_j + a_j \xi_N) \otimes w_j$ Take $\xi_j' = \xi_j + a_j \xi_N$, etc.)

Have $\phi\left(\sum_{j=1}^{N} \xi_j \otimes w_j\right) = 0$ & $\{w_1 \dots w_N\}$ are li.

Then: $\forall v \in V$ $\sum_{j=1}^{N} \phi(\xi_j \otimes w_j)_{(v)} = \sum_{j=1}^{N} \xi_j(v)\, w_j = 0$

$\underset{\{w_1, \dots w_N\}\text{ li}}{\Longrightarrow}$ $\xi_j(v) = 0$ $\forall j = 1 \dots N$

But $\xi_j(v) = 0$ $\forall v \in V$ $\Longrightarrow \xi_j = 0$

Conclude: $\alpha = \sum_{j=1}^{N} 0 \otimes w_j = 0$.

- Claim: If $\dim V = n < \infty$, then ϕ is surjective

Pf/ Let $B_V = \{v_1, \dots, v_n\}$ be a basis for V. Given $f \in \mathrm{Hom}_{\mathbb{K}}(V, W)$, let $w_i = f(v_i)$ $i = 1, \dots n$

Then, $\alpha = \sum_{i=1}^{m} v_i^* \otimes w_i \in V^* \otimes W$ satisfies $\phi(\alpha) = f$ because

$$\sum_{i=1}^{m} \phi(v_i^* \otimes w_i)_{(v_j)} = \sum_{i=1}^{m} \underbrace{v_i^*(v_j)}_{\delta_{ij}} \cdot w_i = w_j$$

So $\phi(\alpha)$ & f agree on B_V, so they are the same function. $\square$