Lecture 36: Canonical tensors & tensor algebra

Last Time: Defined $V \otimes_{\mathbb{K}} W$ of 2 $\mathbb{K}$-vector spaces V & W via universal property

$$\begin{tikzcd} V \times W \arrow[r, "f \text{ bilin}"] \arrow[d, "\varphi", "\text{bil}"'] & W \\ V \otimes_{\mathbb{K}} W \arrow[ur, dashed, "\exists ! \tilde{f} \text{ linear}"'] & \end{tikzcd}$$

$(v,w) \mapsto (v \otimes w)$

Obs: We can define $\bigotimes_{i=1}^{n} V_i = V_1 \otimes_{\mathbb{K}} V_2 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_n$ by working with multilinear forms

$V_1 \times \cdots \times V_n \longrightarrow W$ (linear on each factor, ones values on other factors are fixed)

& an analogous universal property

Prop: $\left.\begin{array}{l} B_V = \{v_i : i \in I\} \text{ basis for } V \\ B_W = \{w_j : j \in J\} \text{ ——— } W \end{array}\right\} \Rightarrow \{v_i \otimes w_j : \begin{smallmatrix} i \in I \\ j \in J \end{smallmatrix}\}$ basis for $V \otimes_{\mathbb{K}} W$

Hom-Tensor adjointness $\exists\ V^* \otimes W \xrightarrow{\Phi} \mathrm{Hom}(V,W)$ with $\Phi(\xi \otimes w) : v \longmapsto \underbrace{\xi(v)}_{\in \mathbb{K}} \cdot w$

(1) Φ linear & injective. , (2) If $\dim V < \infty$, Φ is surjective.

HW12: Many more properties for $\otimes$, including the rank of an element $u \in V \otimes W$ (Problem 17)

Corollary: Let V be finite dimensional $(\dim V = n)$. Choose a basis $B = \{v_i\}_{1 \le i \le n}$ for V and let $\{v_i^*\}_{1 \le i \le n}$ be its dual basis. Then: $\Omega = \sum_{i=1}^{n} v_i^* \otimes v_i$ is independent of the choice of basis. (canonical tensor in $V^* \otimes V$)

$\dim V = n$ $\qquad \Omega = \sum_{i=1}^{n} v_i^* \otimes v_i$ canonical tensor in $V^* \otimes V$

$B = \{v_1, \dots, v_n\}$

Obs 1: If $\dim V$ is infinite, we can't use this to define a canonical tensor
We need to put a topology on $V \otimes V^*$ & take completion.

Obs 2: This also shows Φ in Hom-tensor adjointness cannot be surjective if V is infinite-dimensional.

(Recall: $V^* \otimes W \xrightarrow{\Phi} \mathrm{Hom}(V, W)$ with $\Phi(\xi \otimes w) : v \longmapsto \underbrace{\xi(v)}_{\in \mathbb{K}} \cdot w$)

Take $W = V$, then $\mathrm{id}_V \notin \mathrm{Im}\, \Phi$ (it would correspond to $\Phi(\Omega)$, but we don't have Ω if $\dim V = \infty$ (we can't have infinite sums in $V^* \otimes V$.))

$\boxed{\text{Tensor Algebra}}$

$V = \mathbb{K}$-vector space

Inductively we define $\boxed{T^k(V)} = V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ terms}}$ for all $k \geq 2$

$\rightsquigarrow$ Collect all these tensors into 1 vector space via $\oplus$.

$$\boxed{T^\bullet(V)} = \bigoplus_{n \geq 0} T^n(V) \qquad (T^0(V) = \mathbb{K})$$

Prop: $T^\bullet(V)$ is a $\mathbb{K}$-algebra $=$ $\mathbb{K}$-vector space + ring structure

$$\boxed{T^{\bullet}(V) = \bigoplus_{n \geq 0} T^n(V) \qquad (T^0(V) = \mathbb{K})}$$

<u>Example</u>: $V = \mathbb{K}^n$, then $T^m(V)$

<u>Basis for $T^{\bullet}(V)$</u>: $\{v_i : i \in I\}$ basis for V

Then $\bigcup_{n \geq 0} \underbrace{\{v_{i_1} \otimes \cdots \otimes v_{i_n} : i_j \in I \ \forall j = 1, \ldots, n\}}_{= \text{basis for } T^n(V)}$ is a basis for $T^n(V)$

Symmetric & Exterior Algebras

Next, we define two quotients of $T^k(V)$ that allow us to swap entries in tensors, with/without a sign change. Here, char $\mathbb{K} \neq 2$.

Definition: Fix $k \geq 1$

- $\boxed{S^k(V)} = \mathrm{Sym}^k(V) = V^{\otimes k} \Big/ \left\langle \begin{array}{l} \text{subspace spanned by} \\ v_1 \otimes \cdots \otimes v_k - v_1 \otimes \cdots \otimes \underset{i}{v_{i+1}} \otimes \underset{i+1}{v_i} \otimes \cdots \otimes v_k \\ \quad 1 \leq i \leq n-1 \\ \text{for } v_1, \ldots, v_k \in V \end{array} \right\rangle$

  (k^{th} symmetric product of V)

- Set $S^0(V) = \mathbb{K}$

<u>Obs 1:</u> A typical summand of an element in $S^k(V)$ is written as $v_1 \cdots \cdots v_k$

- $S^k(\mathbb{K}^n) =$

Definition: • $\boxed{\Lambda^k(V)} = V^{\otimes k} \Big/ \left\langle \begin{array}{l} \text{subspace spanned by} \\ v_1\otimes\cdots\otimes v_k + v_1\otimes\cdots\otimes \underset{i}{v_{i+1}}\otimes \underset{i+1}{v_i}\otimes\cdots\otimes v_k \\ \quad 1\le i\le n-1 \\ \text{for } v_1,\dots,v_k\in V \end{array}\right\rangle$ (k^{th} exterior -alternating- product of V)

$(k\geq 2)$

• Set $\Lambda^0(V) = \mathbb{K}$, $\Lambda^1(V) = V$

Lemma: $v_1\otimes\cdots\otimes v_k - sg(\sigma)\, v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)} = 0$ in $\Lambda^k V$ $\quad \forall \sigma\in S_k$, $\forall v_1,\dots v_k\in V$

Obs 2: A typical summand of an element in $\Lambda^k(V)$ is written as $v_1\wedge\cdots\wedge v_k$ (order matters!)

• $v_{\sigma(1)}\wedge\cdots\wedge v_{\sigma(k)} = \text{sign}(\sigma)\,(v_1\wedge\cdots\wedge v_k) \quad \forall \sigma\in S_k.$

• $v_1\wedge\cdots\wedge v_k = 0$ if $v_i = v_j$ for some $i\neq j$

Prop: Basis for $S^k(V)$ & $\Lambda^k(V)$:

(need axiom of choice!)

If $\{v_i : i \in I\}$ is a basis for V & I is totally ordered, then

(1) $\left\{ v_{i_1}^{r_{i_1}} \cdots v_{i_s}^{r_{i_s}} : \begin{array}{l} r_{i_1} + \cdots + r_{i_s} = k \\ s \geq 1, \; r_{i_j} \geq 1 \end{array} \right\}$ is a basis for $S^k(V)$

(2) $\{ v_{i_1} \wedge \cdots \wedge v_{i_k} \quad i_1 < i_2 < \cdots < i_k \text{ in } I \}$ is a basis for $\Lambda^k(V)$

Corollary: Assume $\dim V = n$. Then,

(0) $\dim T^k(V) =$

(1) $\dim S^k V =$

(2) $\dim (\Lambda^k V) =$

We can define Symmetric and exterior algebras:

- $\mathrm{Sym}^\bullet(V) = S^\bullet(V) = \bigoplus_{n \geq 0} S^n(V)$ $\quad (S^0(V) = \mathbb{K}, S^1(V) = V)$
- $\Lambda^\bullet(V) = \bigoplus_{n \geq 0} \Lambda^n(V)$ $\quad (\Lambda^0(V) = \mathbb{K}, \Lambda^1(V) = V)$

Prop: $S^\bullet(V)$ & $\Lambda^\bullet(V)$ are $\mathbb{K}$-algebras. (associative, unital)