Lecture 36: Canonical tensors & tensor algebra

Last time: Defined $V \otimes_{\mathbb{K}} W$ of 2 $\mathbb{K}$-vector spaces V & W via universal property

$$\begin{tikzcd} (v,w) \arrow[d, mapsto] & V \times W \arrow[r, "f \text{ bilin}"] \arrow[d, "\varphi"', "\text{bil}"] \arrow[dr, phantom, "\circlearrowleft"] & W \\ (v\otimes w) & V \otimes_{\mathbb{K}} W \arrow[ur, dashed, "\exists!\, \tilde{f} \text{ linear}"'] & \end{tikzcd}$$

Obs: We can define $\bigotimes_{i=1}^{n} V_i = V_1 \otimes_{\mathbb{K}} V_2 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_n$ by working with multilinear forms

$V_1 \times \cdots \times V_n \longrightarrow W$ (linear on each factor, ones values on other factors are fixed)

& an analogous universal property

(Take $(V_1 \otimes \cdots \otimes V_{n-1}) \otimes V_n$ & induct (see HW12 Problem 15 for $n=3$))

Prop: $\left.\begin{array}{l} B_V = \{v_i : i \in I\} \text{ basis for } V \\ B_W = \{w_j : j \in J\} \text{ —— } W \end{array}\right\} \Rightarrow \{v_i \otimes w_j : \begin{smallmatrix} i \in I \\ j \in J \end{smallmatrix}\}$ basis for $V \otimes_{\mathbb{K}} W$

Hom-Tensor adjointness $\exists\ V^* \otimes W \xrightarrow{\Phi} \operatorname{Hom}(V,W)$ with $\Phi(\xi \otimes w): v \longmapsto \underbrace{\xi(v)}_{\in \mathbb{K}} \cdot w$

(1) Φ linear & injective, (2) If $\dim V < \infty$, Φ is surjective.

HW12: Many more properties for $\otimes$, including the rank of an element $u \in V \otimes W$ (Problem 17)

Corollary: Let V be finite dimensional ($\dim V = n$). Choose a basis $B = \{v_i\}_{1\le i\le n}$ for V and let $\{v_i^*\}_{1\le i\le n}$ be its dual basis. Then: $\sum_{i=1}^{n} v_i^* \otimes v_i$ is independent of the choice of basis. (canonical tensor in $V^* \otimes V$)

Proof: If $\{\tilde{v}_i\}_{1\le i\le n}$ is another basis for V, then

$$\alpha = \sum_{i=1}^{n} v_i^* \otimes v_i \in V^* \otimes V$$
$$\beta = \sum_{i=1}^{n} \tilde{v}_i^* \otimes \tilde{v}_i$$

Claim: $\Phi(\alpha) = \Phi(\beta) = \mathrm{Id}_V \in \mathrm{Hom}(V,V)$ (Φ injective, so $\alpha = \beta$)

Pf/ Write: $v = \sum_{i=1}^{n} a_i v_i = \sum_{j=1}^{n} b_j \tilde{v}_j \in V$

$$\Rightarrow \Phi(\alpha)(v) = \sum_{i=1}^{n} \Phi(v_i^* \otimes v_i)\left(\sum_{j=1}^{n} a_j v_j\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_j \Phi(v_i^* \otimes v_i)(v_j)$$
$$= \sum_{i,j=1}^{n} a_j \underbrace{v_i^*(v_j)}_{\delta_{ij}} \cdot v_i = \sum_{i=1}^{n} a_i v_i = v$$

so $\Phi(\alpha) = \mathrm{id}_V$

Similarly: $\Phi(\beta)(v) = v \quad \forall v$ so $\Phi(\beta) = \mathrm{id}_V$. $\square$

$\dim V = n$ $\qquad \Omega = \sum_{i=1}^{n} v_i^* \otimes v_i$ canonical tensor in $V^* \otimes V$

$B = \{v_1, \dots, v_n\}$

Obs 1: If $\dim V$ is infinite, we can't use this to define a canonical tensor. We need to put a topology on $V \otimes V^*$ & take completion. (to deal with series in $V^* \otimes V$)

Obs 2: This also shows Φ in Hom-tensor adjointness cannot be surjective if V is infinite-dimensional.

(Recall: $V^* \otimes W \xrightarrow{\Phi} \mathrm{Hom}(V, W)$ with $\Phi(\xi \otimes w) : v \longmapsto \underbrace{\xi(v)}_{\in \mathbb{K}} \cdot w$)

Take $W = V$, then $\mathrm{id}_V \notin \mathrm{Im}\,\Phi$ (it would correspond to $\Phi(\Omega)$, but we don't have Ω if $\dim V = \infty$ (we can't have infinite sums in $V^* \otimes V$.))

Tensor Algebra

V = $\mathbb{K}$-vector space

Inductively we define $\boxed{T^k(V)} = V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ terms}}$ for all $k \geq 2$

$\rightsquigarrow$ Collect all these tensors into 1 vector space via $\oplus$.

$$\boxed{T^\bullet(V)} = \bigoplus_{n \geq 0} T^n(V) \qquad (T^0(V) = \mathbb{K})$$

Prop: $T^\bullet(V)$ is a $\mathbb{K}$-algebra $\doteq$ $\mathbb{K}$-vector space + ring structure

Proof: Enough to define the multiplication $T^n(V) \times T^m(V) \longrightarrow T^{n+m}(V)$

via $(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m$ ($\mathbb{K}$-bilinear $\Longrightarrow$ distrib laws)

Well-defined? (1) Use bases & extend bilinearly

(2) Universal Property:

$$\begin{tikzcd}
\overbrace{(V \times \cdots \times V)}^{n \text{ copies}} \times \overbrace{(V \times \cdots \times V)}^{m \text{ copies}} \arrow[r, "\text{``mult''}"] \arrow[d, "(\varphi_n, \varphi_m)"] & T^{n+m}(V) \\
T^n(V) \times T^m(V) \arrow[ur, dashed, "\text{mult}"] \arrow[d, "\varphi"'] & \\
T^n(V) \otimes T^m(V) \arrow[uur] \arrow[ur, phantom, "="] &
\end{tikzcd}$$

v.s.

$\Rightarrow T^n(V) \times T^m(V) \longrightarrow T^{n+m}(V)$

mult $= \varphi$ bilinear in def of $T^n(V) \otimes T^m(V) = T^{n+m}(V)$

$$\boxed{T^{\bullet}(V) = \bigoplus_{n \geq 0} T^n(V) \qquad (T^0(V) = \mathbb{K})}$$

Example: $V = \mathbb{K}^n$, then $T^m(V) \longleftrightarrow$ homogeneous degree m polynomials in n non-commuting variables.

- variables $x_i \longleftrightarrow e_i$ (standard basis for $\mathbb{K}^n$)

Eg. $\left(\sum_j a_i e_i\right) \otimes \left(\sum_j b_j e_j\right) = \sum_{i,j} a_i b_j \, (e_i \otimes e_j)$

$$e_i \otimes e_j \updownarrow x_i \cdot x_j \text{ monomial}$$

$\rightsquigarrow T^2(\mathbb{K}^n) =$ homog degree 2 polys in n non-commuting variables.

Basis for $T^{\bullet}(V)$: $\{v_i : i \in I\}$ basis for V

Then $\bigcup_{n \geq 0} \underbrace{\{v_{i_1} \otimes \cdots \otimes v_{i_n} : i_j \in I \; \forall j = 1, \dots, n\}}_{= \text{basis for } T^n(V)}$ is a basis for $T^{\bullet}(V)$

Symmetric & Exterior Algebras

Next, we define two quotients of $T^k(V)$ that allow us to swap entries on tensors, with/ without a sign change. Here, char $\mathbb{K} \neq 2$.

Definition: Fix $k \geq 1$

- $\boxed{S^k(V)} = \mathrm{Sym}^k(V) = V^{\otimes k} \Big/ \left\langle \begin{array}{l} \text{subspace spanned by} \\ v_1 \otimes \cdots \otimes v_k - v_1 \otimes \cdots \otimes \underset{i}{\underset{\uparrow}{v_{i+1}}} \otimes \underset{i+1}{\underset{\uparrow}{v_i}} \otimes \cdots \otimes v_k \\ \quad 1 \leq i \leq n-1 \\ \text{for } v_1, \ldots, v_k \in V \end{array} \right\rangle$

  (k^{th} symmetric product of V)

- Set $S^0(V) = \mathbb{K}$

Obs 1: A typical summand of an element in $S^k(V)$ is written as $v_1 \cdot \cdots \cdot v_k$

- $S^k(\mathbb{K}^n)$ = deg k homogeneous polynomials in n commuting variables!

Eg: $S^2(\mathbb{K}^n) \ni \left(\sum_{i=1}^n a_i e_i\right) \otimes \left(\sum_{j=1} b_j e_j\right) = \sum_{i,j=1}^n a_i b_j \, e_i \otimes e_j$

$$= \sum_{i<j} (a_i b_j + a_j b_i) \, e_i e_j + \sum_{i=1}^n a_i b_i \, e_i e_i$$

$\downarrow$

$e_i \otimes e_j = e_j \otimes e_i$ in $S^2(\mathbb{K}^n)$

$$\longleftrightarrow \sum_{i<j} (a_i b_j + a_j b_i) \, x_i x_j + \sum_{i=1}^n a_i b_i \, x_i^2 .$$

Definition: • $\boxed{\Lambda^k(V)} = V^{\otimes k} \Big/ \left\langle \begin{array}{l} \text{subspace spanned by} \\ v_1 \otimes \cdots \otimes v_k + v_1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ i}}{v_{i+1}} \otimes \underset{\substack{\uparrow \\ i+1}}{v_i} \otimes \cdots \otimes v_k \\ \quad 1 \le i \le n-1 \\ \text{for } v_1, \dots, v_k \in V \end{array} \right\rangle$ (k^{th} exterior -alternating- product of V)

$(k \geq 2)$

• Set $\Lambda^0(V) = \mathbb{K}$, $\Lambda^1(V) = V$

Lemma: $v_1 \otimes \cdots \otimes v_k - sg(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} = 0$ in $\Lambda^k V$ $\quad \forall \sigma \in S_k$, $\forall v_1, \dots v_k \in V$

Pf/ By induction on $\mathrm{len}(\sigma)$ (write σ as a product of simple transpositions) $(i\ i{+}1)$

Base case $\mathrm{len}(\sigma) = 1$: is the definition of $\Lambda^k(V)$. $\square$

Obs 2: A typical summand of an element in $\Lambda^k(V)$ is written as

$v_1 \wedge \cdots \wedge v_k$ (order matters!)

• $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \mathrm{sign}(\sigma)\, (v_1 \wedge \cdots \wedge v_k) \quad \forall \sigma \in S_k$.

• $v_1 \wedge \cdots \wedge v_k = 0$ if $v_i = v_j$ for some $i \neq j$ (by Lemma)

sign $(i\,j) = -1$, so flipping signs introduces a negative sign

Prop: Basis for $S^k(V)$ & $\Lambda^k(V)$:

If $\{v_i : i \in I\}$ is a basis for V & I is totally ordered, then (need axiom of choice!)

(1) $\left\{ v_{i_1}^{r_{i_1}} \cdots v_{i_s}^{r_{i_s}} : \begin{array}{l} r_{i_1} + \cdots + r_{i_s} = k \\ s \geq 1, \ r_{i_j} \geq 1 \end{array} \right\}$ is a basis for $S^k(V)$

(2) $\{ v_{i_1} \wedge \cdots \wedge v_{i_k} \quad i_1 < i_2 < \cdots < i_k \text{ in } I \}$ is a basis for $\Lambda^k(V)$

Corollary: Assume $\dim V = n$. Then,

(0) $\dim T^k(V) = n^k$ ($\dim V \otimes W = \dim V \cdot \dim W$ + induct on k).

(1) $\dim S^k V = \binom{n+k-1}{k}$ (# monomials of deg k in n variables)

(2) $\dim (\Lambda^k V) = \binom{n}{k}$ ($= 0$ if $k > n$)

We can define symmetric and exterior algebras:

- $\mathrm{Sym}^\bullet(V) = S^\bullet(V) = \bigoplus_{n \geq 0} S^n(V)$ ($S^0(V) = \mathbb{K}$, $S^1(V) = V$)
- $\Lambda^\bullet(V) = \bigoplus_{n \geq 0} \Lambda^n(V)$ ($\Lambda^0(V) = \mathbb{K}$, $\Lambda^1(V) = V$)

Prop: $S^\bullet(V)$ & $\Lambda^\bullet(V)$ are $\mathbb{K}$-algebras. (associative, unital)

Pf/ Follow the same idea as in $T^\bullet(V)$. Define multiplication

$$\Phi: S^n(V) \times S^m(V) \longrightarrow S^{n+m}(V)$$

$$\overline{v_1 \otimes \cdots \otimes v_n} \times \overline{w_1 \otimes \cdots \otimes w_m} \longmapsto \overline{v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_n}$$

$$\Psi: \Lambda^n(V) \times \Lambda^m(V) \longrightarrow \Lambda^{n+m}(V)$$

$$v_1 \wedge \cdots \wedge v_n \times w_1 \wedge \cdots \wedge w_m \longmapsto v_1 \wedge \cdots \wedge v_n \wedge w_1 \wedge \cdots \wedge w_m$$

More precisely: $\Phi(\overline{u}, \overline{u'}) = \overline{\varphi(u, u')}$ in $S^{n+m}(V)$

$\Psi(\overline{u}, \overline{u'}) = \overline{\varphi(u, u')}$ in $\Lambda^{n+m}(V)$ $[\varphi: T^n(V) \times T^m(V) \to T^{n+m}(V)]$

- To show it's well-defined, need to show the relations defining $S^k(V)$ & $\Lambda^k(V)$ are preserved (Exercise)
- Multiplications is associative, distributive by construction. Unit: $1 \in \mathbb{K} = S^0(V) = \Lambda^0(V)$ $\square$