Lecture 37: More on Tensor Algebras, Minors of a matrix

Last Time: Defined $T^\bullet(V)$, $S^\bullet(V)$ & $\Lambda^\bullet(V)$

- $S^k(V) = \mathrm{Sym}^k(V) = V^{\otimes k} \Big/ \left\langle \begin{array}{l} \text{subspace spanned by} \\ v_1\otimes\cdots\otimes v_k - v_1\otimes\cdots\otimes \underset{\uparrow\, i}{v_{i+1}}\otimes \underset{\uparrow\, i+1}{v_i}\otimes\cdots\otimes v_k \\ \quad 1\leq i\leq k-1 \\ \text{for } v_1,\ldots,v_k\in V \end{array}\right\rangle$, $S^0(V) = \mathbb{K}$, $S^1(V) = V$

  (k^{th} symmetric product of V)

  $k\geq 2$

- $\Lambda^k(V) = V^{\otimes k} \Big/ \left\langle \begin{array}{l} \text{subspace spanned by} \\ v_1\otimes\cdots\otimes v_k + v_1\otimes\cdots\otimes \underset{\uparrow\, i}{v_{i+1}}\otimes \underset{\uparrow\, i+1}{v_i}\otimes\cdots\otimes v_k \\ \quad 1\leq i\leq k-1 \\ \text{for } v_1,\ldots,v_k\in V \end{array}\right\rangle$, $\Lambda^0(V) = \mathbb{K}$ $\Lambda^1(V) = V$.

  (k^{th} exterior product of V)

Bases: $\mathcal{B}\{v_i : i\in I\}$ basis for V

For $V^{\otimes n} = \{ v_{i_1}\otimes\cdots\otimes v_{i_n} \quad i_1,\ldots,i_n\in I\}$

For $S^n(V) = \{ v_{i_1}^{r_1}\cdots v_{i_s}^{r_s} \quad i_1<i_2<\cdots<i_s \quad \underset{r_i\geq 0}{r_1+\cdots+r_s} = n\}$

For $\Lambda^n(V) = \{ v_{i_1}\wedge\cdots\wedge v_{i_n} \quad i_1<i_2<\cdots<i_n\}$

Thm: $T^\bullet(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$, $S^\bullet(V) = \bigoplus_{k=0}^{\infty} S^k(V)$, $\Lambda^\bullet(V) = \bigoplus_{k=0}^{\infty}\Lambda^k(V)$ are $\mathbb{K}$-algebras

Pf/ Define mult. $V^{\otimes k}\times V^{\otimes \ell}\longrightarrow V^{\otimes(k+\ell)}$ by concatenation of indecomp tensors. Show it descends to 2 quotients.

Alternative approach: in char$(\mathbb{K})=0$ we can view $S^n(V)$ & $\Lambda^n(V)$ as subspaces of $T^n(V)$. & the multiplication respects the structure.

Prop: (1) $S^2 = S$, $A^2 = A$

(2) $\mathrm{Ker}(S) = \langle v_1 \otimes \cdots \otimes v_n - v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n : 1 \leq i \leq n-1 \quad v_1, \dots v_n \in V \rangle$

$\mathrm{Ker}(A) = \langle v_1 \otimes \cdots \otimes v_n + v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n : 1 \leq i \leq n-1 \quad v_1, \dots v_n \in V \rangle$

$\Rightarrow \mathrm{Im}\, S \simeq S^n(V)$ & $\mathrm{Im}\, A \simeq \Lambda^n(V)$. (HW13)

Obs: View $\mathrm{Sym}^n(V) = \left(T^n(V)\right)^{S_n}$, $\Lambda^n(V) = \left(T^n(V)\right)^{S_n, \varepsilon}$ signed action $\varepsilon = \mathrm{sg}(\sigma)$

Compatibility with linear maps

Prop: $f : V \xrightarrow[\text{linear}]{} W \rightsquigarrow \exists\ T^n(f) : T^n(V) \longrightarrow T^n(W)$ linear

$S^n(f) : S^n(V) \longrightarrow S^n(W)$ linear

$\Lambda^n(f) : \Lambda^n(V) \longrightarrow \Lambda^n(W)$ linear

Universal Properties & Decompositions

Prop Given a (unital, assoc.) $\mathbb{K}$-algebra A & a $\mathbb{K}$-linear map $V \xrightarrow{\varphi} A$, then

- $\exists$ a unique extension : $\overline{\varphi} : T^\bullet(V) \longrightarrow A$. $(\overline{\varphi}|_V = \varphi)$
- If A is a commutative algebra, then $\exists!\ \overline{\varphi} : S^\bullet(V) \longrightarrow A$ $\overline{\varphi}|_V = \varphi$
- If A is skew-commutative, ie $ab = -ba$ $\forall a, b \in A$, then

$$\exists!\ \overline{\varphi}\ \Lambda^\bullet(V) \longrightarrow A. \qquad \overline{\varphi}|_V = \varphi.$$

Prop : $\Phi : V \otimes V \simeq S^2(V) \oplus \Lambda^2(V)$ ($\simeq$ even + odd func)

$\boxed{\text{Q: What happens to } T^n, S^n \ \& \ \Lambda^n \text{ when we consider direct sums?}}$

Lemma: Consider 2 $\mathbb{K}$-vector spaces V & W. Then $\forall n$

(1) $S^n(V \oplus W) = \bigoplus_{i=0}^{n} S^i(V) \otimes S^{n-i}(W)$

(2) $\Lambda^n(V \oplus W) = \bigoplus_{i=0}^{n} \Lambda^i(V) \otimes \Lambda^{n-i}(W)$

(3) $T^n(V \oplus W) = \bigoplus_{k=0}^{n} \left(\bigoplus_{i_1+\dots+i_k=n} \left(T^{i_1}(V) \otimes T^{i_2}(W) \otimes T^{i_3}(V) \otimes \cdots \right) \right)$

Minors of a matrix

$V \simeq \mathbb{K}^n$, $W \cong \mathbb{K}^m$, $V \xrightarrow{f} W$

$B = \{v_1, \dots v_n\}$ $B' = \{w_1, \dots w_m\}$ $(f \in \mathrm{Mat}_{m \times n}(\mathbb{K}))$

- Write
$$\begin{array}{ccc} \Lambda^k(V) & \xrightarrow{\Lambda^k(f)} & \Lambda^k(W) \\ \wr\!\wr & & \wr\!\wr \\ \mathbb{K}^{\binom{n}{k}} & & \mathbb{K}^{\binom{m}{k}} \end{array}$$

Def: Minors of f are the entries of the matrix for g.

More precisely, write $\underline{i} = (i_1, \dots, i_k) \in \binom{[m]}{k}$ (k-subsets of $[m]$)

$\underline{j} = (j_1, \dots, j_k) \in \binom{[n]}{k}$ (—— $[n]$)

$[N] = \{1, \dots, N\}$

$$\Delta^{\underline{i}}_{\underline{j}}(f) = \text{coeff of } \omega_{i_1} \wedge \cdots \wedge \omega_{i_k} \text{ in } \left(\sum_{i=1}^{m} a_{ij_1} \omega_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^{m} a_{ij_k} \omega_i\right)$$

<u>Recall</u> $\omega_{\sigma(i_1)} \wedge \cdots \wedge \omega_{\sigma(i_k)} = \operatorname{sgn}(\sigma)\, \omega_{i_1} \wedge \cdots \wedge \omega_{i_k} \quad \forall \sigma \in S_k.$

$$\boxed{\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{\sigma(1)1}\, a_{\sigma(2)2} \cdots a_{\sigma(n)n}.}$$

Consequence 2 Row-expansion formula for $\det(A)$

Obs: Combine Consequences 1 & 2 to get column expansion formulas.

Consequence 3 A with 2 equal rows, resp. cols, then $\det(A) = 0$.

$\boxed{\text{Cofactor formula}}$

$\mathrm{Cof}(A)_{i,j} = (-1)^{i+j} \det A^{(i,j)}$ A without row i col j

<u>Consequence</u> 4 $(\mathrm{Cof}\,A)^T A = \det(A)\, I_n = A(\mathrm{Cof}\,A)^T$

<u>Obs:</u> Same carries over to matrices $A \in \mathrm{Mat}_{n\times n}(R)$ for R = commutative ring (use permutation formula to define $\det(A)$)

The Cofactor formula yields Cayley–Hamilton for $A \in \mathrm{Mat}_{n\times n}(R)$.

$\boxed{\text{Permanents}}$

Q. What happens if we do this for $S^k(V) \xrightarrow{S^k(f)} S^k(W)$? **A** Permanents!

Def: $\mathrm{Perm}(f)_{\underline{i},\underline{j}}$ = coeff of $w_{i_1}\cdots w_{i_k}$ in $S^k(f)_{(v_{j_1}\cdots v_{j_k})}$ $\left(\begin{array}{l} 1\le i_1<\cdots<i_k\le m \\ 1\le j_1<\cdots<j_k\le n \end{array}\right)$

⚠ We are NOT allowed to have repetitions, so we are not capturing <u>all</u> the coefficients of $S^k(f)$.

In particular, for $n=m=k$, we have $\mathrm{Perm}(A)$ = coeff of $w_1\cdots w_n$ in

$$\left(\sum_{i=1}^n a_{i1}w_1\right)\left(\sum_{i=1}^n a_{i2}w_i\right)\cdots\left(\sum_{i=1}^n a_{in}w_n\right)$$

$$\Longrightarrow \quad \mathrm{Perm}(A) = \sum_{\sigma\in S_n} a_{\sigma(1)1}\cdots a_{\sigma(n)n}$$

⚠ It is no longer true that matrices with repeated rows have vanishing permanent. This makes it very hard to compute! No Algorithms for perm!