Lecture 37: More on Tensor Algebras, Minors of a matrix

Last Time: Defined $T^\bullet(V)$, $S^\bullet(V)$ & $\Lambda^\bullet(V)$

- $S^k(V) = \mathrm{Sym}^k(V) = V^{\otimes k} \Big/ \left\langle \begin{array}{c}\text{subspace spanned by} \\ v_1\otimes\cdots\otimes v_k - v_1\otimes\cdots\otimes v_{i+1}\otimes v_i\otimes\cdots\otimes v_k \\ 1\leq i\leq k-1 \\ \text{for } v_1,\dots,v_k\in V\end{array}\right\rangle$, $S^0(V)=\mathbb{K}$, $S^1(V)=V$

(k^{th} symmetric product of V) $k\geq 2$

$\overline{v_1\otimes\cdots\otimes v_k} = v_1\cdots v_k$

- $\Lambda^k(V) = V^{\otimes k} \Big/ \left\langle \begin{array}{c}\text{subspace spanned by} \\ v_1\otimes\cdots\otimes v_k + v_1\otimes\cdots\otimes v_{i+1}\otimes v_i\otimes\cdots\otimes v_k \\ 1\leq i\leq k-1 \\ \text{for } v_1,\dots,v_k\in V\end{array}\right\rangle$, $\Lambda^0(V)=\mathbb{K}$ $\Lambda^1(V)=V$.

(k^{th} exterior product of V)

$\overline{v_1\otimes\cdots\otimes v_k} = v_1\wedge\cdots\wedge v_k$

$v_{\sigma(1)}\wedge\cdots\wedge v_{\sigma(k)} = \mathrm{sign}(\sigma)\, v_1\wedge\cdots\wedge v_k$ $\forall \sigma\in S_k$

Bases: $B\{v_i : i\in I\}$ basis for V $\quad \dim V = m$

For $V^{\otimes n} = \{v_{i_1}\otimes\cdots\otimes v_{i_n} \quad i_1,\dots,i_n\in I\}$ $\quad \dim = m^n$

For $S^n(V) = \{v_{i_1}^{r_1}\cdots v_{i_s}^{r_s} \quad i_1<i_2<\cdots<i_s \quad r_1+\cdots+r_s = n\}$, $r_i\geq 0$ $\quad \dim = \binom{m+n-1}{n}$

For $\Lambda^n(V) = \{v_{i_1}\wedge\cdots\wedge v_{i_n} \quad i_1<i_2<\cdots<i_n\}$ $\quad \dim = \binom{m}{n}$

Thm: $T^\bullet(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$, $S^\bullet(V) = \bigoplus_{k=0}^{\infty} S^k(V)$, $\Lambda^\bullet(V) = \bigoplus_{k=0}^{\infty}\Lambda^k(V)$ are $\mathbb{K}$-algebras

Pf/ Define mult. $V^{\otimes k}\times V^{\otimes \ell}\longrightarrow V^{\otimes(k+\ell)}$ by concatenation of indecomp tensors. Show it descends to quotients.

Alternative approach: in char$(\mathbb{K})=0$ we can view $S^n(V)$ & $\Lambda^n(V)$ as subspaces of $T^n(V)$. & the multiplication respects the structure.

- Define $S_n \circlearrowright T^n(V)$ via $\begin{cases} \sigma\cdot(v_1\otimes\cdots\otimes v_n) = \mathrm{sg}(\sigma)\, v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)} \\ \sigma\bullet(v_1\otimes\cdots\otimes v_n) = v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)} \end{cases}$

(Need to show it extends from index tensors to $T^n(V)$, can do this via univ property $\sigma_\bullet : \underbrace{V\times\cdots\times V}_{n \text{ times}} \longrightarrow T^n(V)$ multilinear

$\leadsto \exists!\ \overline{\sigma}_\bullet : T^n(V) \longrightarrow T^n(V)$ with $\sigma(v_1,\ldots,v_n) = \overline{\sigma}(v_1\otimes\cdots\otimes v_n)$)

- Define 2 operators $S, A : T^nV \longrightarrow T^nV$ via 2 actions above

$$S(\xi) = \frac{1}{n!}\sum_{\sigma\in S_n}\sigma(\xi) \qquad \& \qquad A(\xi) = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\,\sigma(\xi)$$

Prop: (1) $S^2 = S$, $A^2 = A$

(2) $\mathrm{Ker}(S) = \langle v_1\otimes\cdots\otimes v_n - v_1\otimes\cdots\otimes v_{i+1}\otimes v_i\otimes\cdots\otimes v_n : 1\leq i\leq n-1 \quad v_1,\ldots,v_n\in V\rangle$

$\mathrm{Ker}(A) = \langle v_1\otimes\cdots\otimes v_n + v_1\otimes\cdots\otimes v_{i+1}\otimes v_i\otimes\cdots\otimes v_n : 1\leq i\leq n-1 \quad v_1,\ldots,v_n\in V\rangle$

$\Longrightarrow \mathrm{Im}\, S \simeq S^n(V)$ & $\mathrm{Im}\, A \simeq \Lambda^n(V)$. (HW13)

Obs: View $\mathrm{Sym}^n(V) = (T^n(V))^{S_n}$, $\Lambda^n(V) = (T^n(V))^{S_n,\varepsilon}$ signed action $\varepsilon = \mathrm{sg}(\sigma)$

Compatibility with linear maps

<u>Prop:</u> $f : V \underset{\text{linear}}{\longrightarrow} W \rightsquigarrow \exists\, T^n(f) : T^n(V) \longrightarrow T^n(W)$ linear

$S^n(f) : S^n(V) \longrightarrow S^n(W)$ linear

minors of matrices (Later today) $\rightarrow$ $\Lambda^n(f) : \Lambda^n(V) \longrightarrow \Lambda^n(W)$ linear

<u>Proof:</u>

- $T^n(f)(v_1 \otimes \cdots \otimes v_n) = f(v_1) \otimes \cdots \otimes f(v_n)$ determines $T^n(f)$ uniquely

(Via univ prop:

$$\begin{tikzcd}
V \times \cdots \times V \arrow[rr, "f \times \cdots \times f", "\text{mult}"'] \arrow[d, "\varphi"'] & & W \times \cdots \times W \arrow[r, "\varphi", "\text{mult}"'] & T^n(W) \\
T^n(V) \arrow[rrru, dashed, "\exists!\, T^n(f)", "\text{linear}"'] \arrow[rr, phantom, "\circlearrowleft"] & & &
\end{tikzcd}$$

)

- $S^n(f)(v_1 \cdots v_n) = f(v_1) \cdots f(v_n)$ defines $S^n(f)$
- $\Lambda^n(f)(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n)$ —— $\Lambda^n(f)$

Check $T^n(f)\big|_{\ker S} \subset \operatorname{Ker}(S)$, $\Lambda^n(f)\big|_{\operatorname{Ker}(A)} \subset \operatorname{Ker}(A)$

Universal Properties & Decompositions

<u>Prop</u> Given a (unital, assoc.) $\mathbb{K}$-algebra A & a $\mathbb{K}$-linear map $V \xrightarrow{\varphi} A$, then

- $\exists$ a unique extension : $\overline{\varphi} : T^\bullet(V) \longrightarrow A$. $\left(\overline{\varphi}|_V = \varphi\right)$
- If A is a commutative algebra, then $\exists!\ \overline{\varphi} : S^\bullet(V) \longrightarrow A$ $\quad \overline{\varphi}|_V = \varphi$
- If A is skew-commutative, ie $ab = -ba$ $\forall a, b \in A$, then

$$\exists!\ \overline{\varphi}\ \ \Lambda^\bullet(V) \longrightarrow A. \qquad \overline{\varphi}|_V = \varphi.$$

<u>Prop</u>: $\Phi : V \otimes V \simeq S^2(V) \oplus \Lambda^2(V)$ (≃ even + odd func)

$$(v \otimes v') \longmapsto \left(\frac{v \otimes v' + v' \otimes v}{2} , \frac{v \otimes v' - v' \otimes v}{2} \right)$$

⚠ Fails for $n > 2$
$V^{\otimes n} = \bigoplus_{\lambda \vdash n} S^\lambda(V)$

Pf/ Well-defined via universal property:

Write
$$\begin{aligned} V \times V &\longrightarrow S^2(V) \oplus \Lambda^2(V) \quad \text{bilinear} \\ (v, v') &\longmapsto (v \cdot v' , v \wedge v') \end{aligned}$$

$\Rightarrow$ This map factors through $V \otimes V$. This defines Φ.

We view $S^2(V)$ & $\Lambda^2(V)$ as subspaces of $\Lambda \otimes \Lambda$. & construct the inverse map Φ^{-1} via the inclusions $S^2(V) \hookrightarrow T^2(V)$ & $\Lambda^2(V) \longrightarrow T^2(V)$ $\square$

$\boxed{\text{Q: What happens to } T^n, S^n \text{ \& } \Lambda^n \text{ when we consider direct sums?}}$

Lemma: Consider 2 $\mathbb{K}$-vector spaces V & W. Then $\forall n$

(1) $S^n(V \oplus W) = \bigoplus_{i=0}^{n} S^i(V) \otimes S^{n-i}(W)$

(Think of polynomials in commuting variables x_i (for basis elements in V)
y_j ———————)

(2) $\Lambda^n(V \oplus W) = \bigoplus_{i=0}^{n} \Lambda^i(V) \otimes \Lambda^{n-i}(W)$

(3) $T^n(V \oplus W) = \bigoplus_{k=0}^{n} \left(\bigoplus_{i_1+\dots+i_k=n} \left(T^{i_1}(V) \otimes T^{i_2}(W) \otimes T^{i_3}(V) \otimes \dots \right) \right)$

(Variables don't commute, so we can't rearrange putting V-part before W-piece)

Pf/ Pick bases for V & W & check both sides of each identity share the same natural bases.

Minors of a matrix

$V \simeq \mathbb{K}^n$, $\quad W \cong \mathbb{K}^m$, $\quad V \xrightarrow{f} W$

$B = \{v_1, \dots, v_n\}$ $\quad B' = \{w_1, \dots, w_m\}$ $\quad (f \in \mathrm{Mat}_{m \times n}(\mathbb{K}))$

- Write

$$\begin{tikzcd} \Lambda^k(V) \arrow[r, "\Lambda^k(f)"] \arrow[d, phantom, "\wr\|"] \arrow[dr, phantom, "\circlearrowleft"] & \Lambda^k(W) \arrow[d, phantom, "\wr\|"] \\ \mathbb{K}^{\binom{n}{k}} \arrow[r, dashed, "g \in \mathrm{Mat}_{\binom{m}{k} \times \binom{n}{k}}(\mathbb{K})"'] & \mathbb{K}^{\binom{m}{k}} \end{tikzcd}$$

<u>Bases</u>: $\left\{ v_{j_1} \wedge \cdots \wedge v_{j_k} : 1 \leq j_1 < \cdots < j_k \leq n \right\}$ for $\Lambda^k(V)$ $(= \emptyset$ if $k > n)$

$\left\{ w_{i_1} \wedge \cdots \wedge w_{i_k} : 1 \leq i_1 < \cdots < i_k \leq m \right\}$ — $\Lambda^k(W)$ $(= \emptyset$ if $k > m)$

<u>Def</u>: <u>Minors</u> of f are the entries of the matrix for g.

More precisely, write $\underline{i} = (i_1, \dots, i_k) \in \binom{[m]}{k}$ (k-subsets of $[m]$) $\quad [N] = \{1, \dots, N\}$

$\underline{j} = (j_1, \dots, j_k) \in \binom{[m]}{k}$ (——— $[n]$)

$$\Rightarrow \quad \Lambda^k(f)_{\underline{i}, \underline{j}} = \det\left(f^{\underline{i}}_{\underline{j}} \right) = \boxed{\Delta^{\underline{i}}_{\underline{j}}(f)}$$

$\hookrightarrow$ submatrix of f with rows in $\underline{i}$ & columns in $\underline{j}$

Q How do we compute $\Lambda^k(f)_{\underline{i}, \underline{j}}$? Write $f(v_j) = \sum_{i=1}^{m} a_{ij} w_i$

$$\Lambda^k(f)(v_{j_1} \wedge \cdots \wedge v_{j_k}) = \left(\sum_{i=1}^{m} a_{i j_1} w_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^{m} a_{i j_k} w_i\right)$$

$$\Rightarrow \Delta^{\underline{i}}_{\underline{j}}(f) = \text{coeff of } w_{i_1} \wedge \cdots \wedge w_{i_k} \text{ in } \Lambda^k(f)(v_{j_1} \wedge \cdots \wedge v_{j_k})$$

$\Delta^{\underline{i}}_{\underline{j}}(f)$ = coeff of $w_{i_1}\wedge\cdots\wedge w_{i_k}$ in $\left(\sum_{i=1}^m a_{ij_1} w_i\right)\wedge\cdots\wedge\left(\sum_{i=1}^m a_{ij_k} w_i\right)$

Recall $w_{\sigma(i_1)}\wedge\cdots\wedge w_{\sigma(i_k)} = \operatorname{sign}(\sigma)\, w_{i_1}\wedge\cdots\wedge w_{i_k}$ $\forall\, \sigma\in S_k$.

So coeff of $w_{i_1}\wedge\cdots w_{i_k}$ in $\left(\sum_{i=1}^m a_{ij_1} w_i\right)\wedge\cdots\wedge\left(\sum_{i=1}^m a_{ij_k} w_i\right)$ is

$$\Delta^{\underline{i}}_{\underline{j}}(f) = \sum_{\sigma\in S_k} \operatorname{sign}(\sigma)\, a_{i_{\sigma(1)} j_1}\, a_{i_{\sigma(2)} j_2}\cdots a_{i_{\sigma(k)} j_k}$$

In particular: $k=n=m$ write $\det(f) = \Delta^{1,\dots,n}_{1,\dots,n}(f)$

This recovers the permutation formula for determinants.

$$\det(A) = \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\, a_{\sigma(1)1}\, a_{\sigma(2)2}\cdots a_{\sigma(n)n}.$$

$$\underset{\tau=\sigma^{-1}}{=} \sum_{\tau\in S_n} \operatorname{sign}(\tau)\, a_{1\tau(1)}\, a_{2\tau(2)}\cdots a_{n\tau(n)}$$

Consequence ① $\det(A) = \det(A^T)$ $A\in \mathrm{Mat}_{n\times n}(\mathbb{K})$

$$\boxed{\det(A) = \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\, a_{\sigma(1)1}\, a_{\sigma(2)2} \cdots a_{\sigma(n)n}.}$$

Consequence 2 Row-expansion formula for $\det(A)$

Pf/ Fix $i \Rightarrow \det(A) = \sum_{j=1}^{n} a_{ij} \underbrace{\sum_{\sigma:\, \sigma(j)=i} \operatorname{sign}(\sigma)\, a_{\sigma(1)1} \cdots \widehat{a_{\sigma(j),j}} \cdots a_{\sigma(n),n}}_{(**)}$

Restrict σ to $\tilde{\sigma} = \{1,\dots,\hat{j},\dots n\} \xrightarrow[b_{ij}]{} \{1,\dots,\hat{i},\dots,n\}$ so $\tilde{\sigma} \in S_{n-1}$ &

$\operatorname{sign}(\sigma) = (-1)^{i+j} \operatorname{sign}(\tilde{\sigma})$

Then $(**) = \det(A^{(i,j)})\,(-1)^{i+j}$ $\Rightarrow \det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A^{(i,j)}$.

(expansion along i^{th} row)

$\square$

Obs: Combine Consequences 1 & 2 to get column expansion formulas.

Consequence 3 A with 2 equal rows, resp. cols, then $\det(A) = 0$.

Pf/ $\det(F) = \Lambda^n(F)(v_1 \wedge \dots \wedge v_n) = 0$ in $\Lambda^n(\mathbb{K}^n) \simeq \mathbb{K}$ by $(*)$ $\square$

Cofactor formula

$\mathrm{Cof}(A)_{i,j} = (-1)^{i+j} \det A^{(i,j)}$ ← A without row i col j

Consequence ④ $(\mathrm{Cof}\,A)^T A = \det(A) I_n = A(\mathrm{Cof}\,A)^T$

Pf/ $\left((\mathrm{Cof}\,A)^T A\right)_{ij} = \sum_{l=1}^{n} (\mathrm{Cof}\,A)^T_{il}\, a_{lj} = \sum_{l=1}^{n} (-1)^{i+l} \det(A^{(l,i)})\, a_{lj}$

- If $i=j$ This is j^{th} column expansion of $\det(A)$.
- If $i \neq j$ ———————— $\det(A')$ where A' is the matrix

obtained from A by replacing i^{th} col of A by the j^{th} col of A

By Consequence ③, $\det(A') = 0$.

Conclude: $(\mathrm{Cof}\,A)^T A = \det A\, I_n$.

- $A(\mathrm{Cof}\,A)^T = \left((\mathrm{Cof}\,A)\, A^T\right)^T = \left((\mathrm{Cof}\,A^T)^T A^T\right)^T = (\det A^T\, I_n)^T = \det A\, I_n$

Obs: Same carries over to matrices $A \in \mathrm{Mat}_{n\times n}(R)$ for R = commutative ring
(use permutation formula to define $\det(A)$)
The Cofactor formula yields Cayley–Hamilton for $A \in \mathrm{Mat}_{n\times n}(R)$.

Permanents

Q: What happens if we do this for $S^k(V) \xrightarrow{S^k(f)} S^k(W)$? **A** Permanents!

Def: $\mathrm{Perm}(f)_{\underline{i},\underline{j}}$ = coeff of $w_{i_1} \cdots w_{i_k}$ in $S^k(f)_{(v_{j_1} \cdots v_{j_k})}$ $\left(\begin{array}{l} 1 \leq i_1 < \cdots < i_k \leq m \\ 1 \leq j_1 < \cdots < j_k \leq n \end{array}\right)$

⚠ We are NOT allowed to have repetitions, so we are not capturing all the coefficients of $S^k(f)$.

In particular, for $n=m=k$, we have $\mathrm{Perm}(A)$ = coeff of $w_1 \cdots w_n$ in

$$\left(\sum_{i=1}^{n} a_{i1} w_i\right)\left(\sum_{i=1}^{n} a_{i2} w_i\right) \cdots \left(\sum_{i=1}^{n} a_{in} w_n\right)$$

$$\Longrightarrow \quad \mathrm{Perm}(A) = \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

⚠ It is no longer true that matrices with repeated rows have vanishing permanent. This makes it very hard to compute! No Algorithms for perm!