Lecture 38: Determinants, Gaussian decompositions, bilinear forms

Recall: $f: V \longrightarrow W$ $\dim V = n$, $\dim W = m$ $B = \{v_1, \dots v_n\}$
$B' = \{w_1, \dots w_m\}$

(i,j) minor: $\Delta^{\underline{i}}_{\underline{j}}(f) = $ coeff of $w_{i_1} \wedge \dots \wedge w_{i_k}$ in $\Lambda^k(f)(v_{j_1} \wedge \dots \wedge v_{j_k})$

$\underline{i} = \{i_1, \dots i_k\} \in \binom{[m]}{k}$ $\underline{j} = \{j_1, \dots, j_k\} \in \binom{[n]}{k}$

- For $n = m = k$ $\Lambda^n(f): \Lambda^n(V) \longrightarrow \Lambda^n(W)$

Prop: $\det(g \circ f) = \det(g) \cdot \det(f)$ (See HW13 Problem 11)

Proof Main idea: $f: V \longrightarrow U$ & $g: U \longrightarrow W$ $\dim V = \dim U = \dim W = n$

Gaussian Decomposition

Def (HW 13 Pr 14): We say $X \in GL_n(\mathbb{K})$ admits a <u>Gaussian Decomposition</u> if

$$X = X^- X^0 X^+$$

where X^0 = invertible diagonal matrix, $X^- = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, $X^+ = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$

Theorem 1: Gaussian decompositions are unique. (whenever they exist)

Theorem 2: X admits G.D $\iff \Delta^{1,\dots,i}_{1,\dots,i}(X) \neq 0 \quad \forall i = 1, \dots, n$ (non-vanishing ppal minors)

(HW13 Pr 14: $n=2$ case)

Pf/

Obs: The construction is also true for $GL_n(R)$ where R is not necessarily a commutative ring (define dets via column expansion).

Obs 2: Non vanishing minors is an open condition in $\mathrm{Mat}_{n\times n}(\mathbb{K})$
We have a dense open set $\mathcal{U} \subseteq \mathrm{Mat}_{n\times n}(\mathbb{K})$ which we parameterize as

(Big Bruhat cell) $\mathcal{U} \cong \underset{(\text{entries in } X^-)}{\mathbb{K}^{\frac{n(n-1)}{2}}} \times \underset{(\text{entries in } X^0)}{(\mathbb{K}^*)^n} \times \underset{(\text{entries in } X^+)}{\mathbb{K}^{\frac{n(n-1)}{2}}}$

Can prove statements on $\mathrm{Mat}_{n\times n}(\mathbb{K})$ by restricting to $\mathcal{U}$.

Ex: Borel-Weil-Bott Theorem

Construct sections on $GL_n(\mathbb{C})$ valued in a line bundle L_λ.

Bilinear forms

Fix V_1, V_2, W $\mathbb{K}$-vector spaces

Recall: $\mathrm{Bil}_{\mathbb{K}}(V_1, V_2, W) = \{ f : V_1 \times V_2 \longrightarrow W \text{ bilinear} \}$ $\quad$ $f(v_1, -) \in \mathrm{Hom}_{\mathbb{K}}(V_2, W)$, $f(-, v_2) \in \mathrm{Hom}_{\mathbb{K}}(V_1, W)$

$$\mathrm{Bil}_{\mathbb{K}}(V_1, V_2, W) = \mathrm{Hom}_{\mathbb{K}}(V_1 \otimes V_2, W)$$

Def: A bilinear form on $V_1 \times V_2$ is an element of $\mathrm{Bil}(V_1, V_2, \mathbb{K})$

Def A bilinear form f on $V_1 \times V_2$ is non-degenerate if

$$\begin{aligned} V_1 &\hookrightarrow V_2^* & \quad \& \quad V_2 &\hookrightarrow V_1^* \\ v_1 &\longmapsto f(v_1, -) & v_2 &\longmapsto f(-, v_2). \end{aligned} \qquad (*)$$

Consequence: If V_1 or V_2 are finite-dimensional & $f \in \mathrm{Bil}(V_1, V_2, \mathbb{K})$ is non degenerate, then $\dim V_1 = \dim V_2 < \infty$

Motivation : Poincaré Duality

Pick X smooth compact manifold of dim $= n$

(V1) $$H_k(X) \otimes H^k(X) \xrightarrow{\int} \mathbb{R} \quad \text{is non-deg.} \quad \forall\, 0 \le k \le n$$

$$\sum_{i=1}^{m} a_i S_i \otimes \sum_{j=1}^{\ell} b_j \psi_j \longmapsto \sum_{i,j} a_i b_j \int_{S_i} \psi_i \qquad a_i, b_j \in \mathbb{R}$$

$S_i = k\text{-cell}$ (simplicial) : $\Delta_k \hookrightarrow S_i \subseteq X$

ψ_i : k-form on X (exterior) $\qquad \int_{S_i} \psi_i = \int_{\Delta_k} S^* \psi_i$ (Riemann integral)

(V2) $$H^{n-k}(X) \otimes H^k(X) \longrightarrow \mathbb{R} \quad \text{is non-deg} \quad \forall\, 0 \le k \le n$$

$$\psi \otimes \xi \longmapsto \int_X \psi \wedge \xi$$

Poincaré duality: $$H_k(X) \underset{(V1)}{\simeq} \left(H^k(X)\right)^* \underset{(V2)}{\simeq} H^{n-k}(X) \qquad \forall\, 0 \le k \le n$$

• Assume $\dim V_1 = \dim V_2 = n$ & pick $B_1 = \{v_1, \dots, v_n\}$ basis for V_1
$B_2 = \{w_1, \dots, w_n\}$ —— V_2

Write $f \in \mathrm{Bil}(V_1, V_2, \mathbb{K})$ via $\langle v_i, w_j \rangle = f(v_i, w_j)$
& build an $n \times n$ matrix $\boxed{Q}$ with $Q_{i,j} = \langle v_i, w_j \rangle \ \forall i, j$

<u>Prop</u>: f is non-degenerate if & only if Q is invertible.

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