Lecture 38: Determinants, Gaussian decompositions, bilinear forms

Recall: $f: V \longrightarrow W$ $\quad \dim V = n, \ \dim W = m \quad B = \{v_1, \dots, v_n\}$
$B' = \{w_1, \dots, w_m\}$

$(\underline{i}, \underline{j})$ minor: $\Delta^{\underline{i}}_{\underline{j}}(f) = \text{coeff of } w_{i_1} \wedge \cdots \wedge w_{i_k} \text{ in } \Lambda^k(f)(v_{j_1} \wedge \cdots \wedge v_{j_k})$

$\underline{i} = \{i_1, \dots, i_k\} \in \binom{[m]}{k} \qquad \underline{j} = \{j_1, \dots, j_k\} \in \binom{[n]}{k}$

- **For $n = m = k$**

$$\begin{tikzcd}
\Lambda^n(V) \arrow[r, "\Lambda^n(f)"] \arrow[d, equal] & \Lambda^n(W) \arrow[d, equal] \\
\mathrm{Sp}(v_1 \wedge \cdots \wedge v_n) \arrow[d, "\cong"] & \mathrm{Sp}(w_1 \wedge \cdots \wedge w_n) \arrow[d, "\cong"] \\
\mathbb{K} \arrow[r, dashed, "\det(f)\cdot"] & \mathbb{K}
\end{tikzcd}$$

$\det(f) = \det([f]_{BB'})$ $\qquad$ (HW 13 Problem 10)

Prop: $\det(g \circ f) = \det(g) \cdot \det(f)$ $\qquad$ (See HW 13 Problem 11)

Proof Main idea: $f: V \longrightarrow U$ & $g: U \longrightarrow W$ $\quad \dim V = \dim U = \dim W = n$

(FUNCTORIALITY)

$$\begin{tikzcd}
\Lambda^n(V) \arrow[r, "\Lambda^n(f)"] \arrow[rr, bend left, "\Lambda^n(g \circ f)"] \arrow[d, "\cong"] & \Lambda^n(U) \arrow[r, "\Lambda^n(g)"] \arrow[d, "\cong"] & \Lambda^n(W) \arrow[d, "\cong"] \\
\mathbb{K} \arrow[r, "\det(f)"] \arrow[rr, bend right, "\det(g \circ f)"'] & \mathbb{K} \arrow[r, "\det(g)"] & \mathbb{K}
\end{tikzcd}$$

$\Rightarrow \boxed{\det(g \circ f) = \det g \cdot \det f.}$

Gaussian Decomposition

Def: We say $X \in GL_n(\mathbb{K})$ admits a <u>Gaussian Decomposition</u> if
(HW 13 Pr 14)

$$X = X^- X^0 X^+$$

where X^0 = invertible diagonal matrix, $X^- = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, $X^+ = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$

(subgroup of $GL_n(\mathbb{K})$) (subgroup of $GL_n(\mathbb{K})$) (subgroup of $GL_n(\mathbb{K})$)

<u>Theorem 1</u>: Gaussian decompositions are unique. (whenever they exist)

Pf/ $X^- X^0 X^+ = Y^- Y^0 Y^+$

$$\underbrace{(Y^-)^{-1} X^-}_{=\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}} \; X^0 \; \underbrace{(X^+)(Y^+)^{-1}}_{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}} = Y^0$$

Gaussian decomp of Y^0 $\Rightarrow$ enough to show uniqueness for diagonal matrices

<u>Claim</u>: $X^- X^0 X^+ = Y^0$ diagonal $\Longrightarrow X^- = X^+ = I_n$

Pf/ $\underbrace{X^0 X^+}_{\text{upper } \Delta} = \underbrace{(X^-) Y^0}_{\text{lower } \Delta}$ so both are diagonal (& invertible!)

$X^0 = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \Rightarrow X^0 X^+ = \begin{pmatrix} x_1 & * & \cdots & * \\ & x_2 & & \vdots \\ & & \ddots & * \\ 0 & & & x_n \end{pmatrix} = \begin{pmatrix} y_1 & & & 0 \\ * & \ddots & & \\ \vdots & & \ddots & \\ * & \cdots & * & y_n \end{pmatrix}$

$Y^0 = \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_n \end{pmatrix}$

forces $x_1 = y_1$, $x_2 = y_2$, ..., $x_n = y_n$

$$0 = (X^0 X^+)_{ij} = \sum_{k=1}^{n} x^0_{ik} x^+_{kj} = \underset{\neq 0}{x^0_{ii}}\, x^+_{ij}$$
$j > i$

$\Rightarrow X^+ = I_n$. <u>Same idea</u>: $X^- = I_n$

Theorem 2: X admits G.D $\iff \Delta^{1,\dots,i}_{1,\dots,i}(X) \neq 0 \ \forall i = 1,\dots,n$ (non-vanishing ppal minors)

(HW13 Pr 14: $n=2$ case)

Pf/ ($\Rightarrow$) $\Delta^{1,\dots,k}_{1,\dots,k}(X) = \Delta^{1,\dots,k}_{1,\dots,k}\left(X^- X^0 X^+\right)$

$$= \Delta^{1,\dots,k}_{1,\dots,k}\left(\begin{array}{cc} & k \\ k & \left(\begin{array}{c|c} \begin{matrix} 1 & \ddots & * \\ 0 & & 1 \end{matrix} & * \\ \hline 0 & \begin{matrix} 1 & \ddots & * \\ 0 & & 1 \end{matrix} \end{array}\right) \end{array} \begin{array}{cc} & k \\ k & \left(\begin{array}{c|c} \begin{matrix} d_1 & \ddots & 0 \\ 0 & & d_k \end{matrix} & 0 \\ \hline 0 & \begin{matrix} d_{k+1} & \ddots & 0 \\ 0 & & d_n \end{matrix} \end{array}\right) \end{array} \begin{array}{cc} & k \\ k & \left(\begin{array}{c|c} \begin{matrix} 1 & \ddots & 0 \\ * & & 1 \end{matrix} & 0 \\ \hline * & \begin{matrix} 1 & \ddots & 0 \\ * & & 1 \end{matrix} \end{array}\right) \end{array} \right) = \det\begin{pmatrix} d_1 & \ddots & 0 \\ 0 & & d_k \end{pmatrix} = d_1 \cdots d_k$$

So $\det(X) = d_1 \cdots d_n \neq 0 \iff$ all $d_i \neq 0$.

$\Rightarrow \Delta^{1,\dots,k}_{1,\dots,k}(\Delta) = d_1 \cdots d_k$ nonzero $\forall k$.

Moreover: $d_1 = \Delta^1_1(X)$

$d_1 d_2 = \Delta^{1,2}_{1,2}(\Delta) \rightsquigarrow d_2 = \dfrac{\Delta^{1,2}_{1,2}(X)}{\Delta^1_1(X)}$ $\Rightarrow$ In general $d_k = \dfrac{\Delta^{1,\dots,k}_{1,\dots,k}(X)}{\Delta^{1,\dots,k-1}_{1,\dots,k-1}(X)}$ $\forall k$.

($\Leftarrow$) Explicitly, write $(X^0)_{11} = \Delta^1_1(X)$, $(X^0)_{kk} = \dfrac{\Delta^{1,\dots,k}_{1,\dots,k}(X)}{\Delta^{1,\dots,k-1}_{1,\dots,k-1}(X)}$

Set $(X^-)_{ij} = \dfrac{\Delta^{1,\dots,i-1,j}_{1,\dots,i}(X)}{\Delta^{1,\dots,i}_{1,\dots,i}(X)}$, $(X^+)_{ji} = \dfrac{\Delta^{1,\dots,i}_{1,\dots,i-1,j}(X)}{\Delta^{1,\dots,i}_{1,\dots,i}(X)}$ $\quad i \leq j$

and check $X = X^- X^0 X^+$ (by induction on n) $\square$

Obs: The construction is also true for $GL_n(R)$ where R is not necessarily a commutative ring (define dets via column expansion).
(c-determinants)

Obs 2: Non vanishing minors is an open condition in $Mat_{n\times n}(\mathbb{K})$
We have a dense open set $\mathcal{U} \subseteq Mat_{n\times n}(\mathbb{K})$ which we parameterize as

(Big Bruhat cell) $$\mathcal{U} \cong \underset{(\text{entries in } X^-)}{\mathbb{K}^{\frac{n(n-1)}{2}}} \times \underset{(\text{entries in } X^0)}{(\mathbb{K}^*)^n} \times \underset{(\text{entries in } X^+)}{\mathbb{K}^{\frac{n(n-1)}{2}}}$$

Can prove statements on $Mat_{n\times n}(\mathbb{K})$ by restricting to $\mathcal{U}$.

Ex: Borel-Weil-Bott Theorem

Construct sections on $GL_n(\mathbb{C})$ valued in a line bundle L_λ.

Bilinear forms

Fix V_1, V_2, W $\mathbb{K}$-vector spaces

Recall: $\mathrm{Bil}_{\mathbb{K}}(V_1, V_2, W) = \{ f : V_1 \times V_2 \longrightarrow W \text{ bilinear} \}$ $\qquad f(v_1, -) \in \mathrm{Hom}_{\mathbb{K}}(V_2, W)$, $f(-, v_2) \in \mathrm{Hom}_{\mathbb{K}}(V_1, W)$

$$\mathrm{Bil}_{\mathbb{K}}(V_1, V_2, W) = \mathrm{Hom}_{\mathbb{K}}(V_1 \otimes V_2, W)$$

Def: A <u>bilinear form</u> on $V_1 \times V_2$ is an element of $\mathrm{Bil}(V_1, V_2, \mathbb{K})$

Def A bilinear form f on $V_1 \times V_2$ is <u>non-degenerate</u> if

$$\begin{aligned} V_1 &\hookrightarrow V_2^* & \quad \& \quad V_2 &\hookrightarrow V_1^* \\ v_1 &\longmapsto f(v_1, -) & v_2 &\longmapsto f(-, v_2). \end{aligned} \qquad (*)$$

Consequence: If V_1 or V_2 are finite-dimensional & $f \in \mathrm{Bil}(V_1, V_2, \mathbb{K})$ is non degenerate, then $\dim V_1 = \dim V_2 < \infty$

Pf/ If $\dim V_1 = n < \infty$, then $\dim V_2 \leq \dim V_1^* = \dim V_1$ (via $V_2 \hookrightarrow V_1^*$)

& $\dim V_1 \leq \dim V_2^* = \dim V_2$ (via $V_1 \hookrightarrow V_2^*$)

So $\dim V_1 = \dim V_2$ & both inclusions in $(*)$ are isomorphisms

Motivation : Poincaré Duality

Pick X smooth compact manifold of dim $= n$

(V1) $\quad H_k(X) \otimes H^k(X) \xrightarrow{\int} \mathbb{R}$ is non-deg. $\forall\, 0 \leq k \leq n$

$$\sum_{i=1}^{m} a_i S_i \otimes \sum_{j=1}^{\ell} b_j \psi_j \longmapsto \sum_{i,j} a_i b_j \int_{S_i} \psi_i \qquad a_i, b_j \in \mathbb{R}$$

S_i = k-cell (simplicial) : $\Delta_k \hookrightarrow S_i \subseteq X$

ψ_i : k-form on X (exterior) $\quad \int_{S_i} \psi_i = \int_{\Delta_k} S^* \psi_i$ (Riemann integral)

(V2) $\quad H^{n-k}(X) \otimes H^k(X) \longrightarrow \mathbb{R}$ is non-deg $\forall\, 0 \leq k \leq n$

$$\psi \otimes \xi \longmapsto \int_X \psi \wedge \xi$$

Poincaré duality: $\quad H_k(X) \underset{(V1)}{\simeq} \left(H^k(X)\right)^* \underset{(V2)}{\simeq} H^{n-k}(X) \qquad \forall\, 0 \leq k \leq n$

• Assume $\dim V_1 = \dim V_2 = n$ & pick $B_1 = \{v_1, \ldots, v_n\}$ basis for V_1
$B_2 = \{w_1, \ldots, w_n\}$ —— V_2

Write $f \in \mathrm{Bil}(V_1, V_2, \mathbb{K})$ via $\langle v_i, w_j \rangle = f(v_i, w_j)$

& build an $n \times n$ matrix $\boxed{Q}$ with $Q_{i,j} = \langle v_i, w_j \rangle \ \forall i,j$

Prop: f is non-degenerate if & only if Q is invertible.

Pf/ ($\Rightarrow$)
$$V_1 \xrightarrow{\varphi_1} V_2^*$$
$$v \longmapsto \{w \mapsto f(v,w)\}$$

So $\varphi_1(v_j)(w_i) = f(v_j, w_i) = Q_{ji}$ $\Longrightarrow$ $\varphi_1(v_j) = \sum Q_{ji} w_i^*$

This means $\varphi_1(v_j) = \sum_{i=1}^{n} Q_{ji} w_i^*$ $\quad [\varphi_1]_{B_1 B_2^*} = Q^T$

Since φ_1 is inj & $\dim V_1 = \dim V_2^* = n < \infty$, we conclude φ_1 is an iso $\Longrightarrow Q^T$ & $Q \in GL_n(\mathbb{K})$

Similarly $V_2 \xrightarrow{\varphi_2} V_1^*$ $\quad \Rightarrow \varphi_2(w_j) = \sum_{i=1}^{n} Q_{ij} v_i^*$

$w_j \longmapsto (v_i \mapsto f(v_i, w_j) = Q_{ij})$ $\quad \Leftrightarrow [\varphi_2]_{B_2 B_1^*} = Q$

($\Leftarrow$) Q invertible $\Rightarrow$ φ_1 & φ_2 are invertible since $[\varphi_1]_{B_1 B_2^*} = Q^T$
$[\varphi_2]_{B_2 B_1^*} = Q$.

$\Longrightarrow f$ is non-degenerate.