Lecture 39: Bilinear forms II (Symmetric forms)

Last Time: $\mathrm{Bil}_{\mathbb{K}}(V_1, V_2, \mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}(V_1 \otimes V_2, \mathbb{K}) = (V_1 \otimes V_2)^*$

- $f \in \mathrm{Bil}_{\mathbb{K}}(V_1, V_2, \mathbb{K})$ non deg $\iff$ induces linear injections

$$\begin{array}{ccc} V_1 & \overset{\varphi_1}{\hookrightarrow} & V_2^* \\ v & \longmapsto & f(v,-) \end{array} \qquad \begin{array}{ccc} V_2 & \overset{\varphi_2}{\hookrightarrow} & V_1^* \\ v & \longmapsto & f(-,v) \end{array}$$

Prop: V_1, V_2 finite dimensional, f non-deg $\underset{?}{\overset{\Longrightarrow}{\Longleftarrow}}$ $\dim V_1 = \dim V_2$

A: If $V_1 \simeq \mathbb{K}^n \simeq V_2$ write $B_1 = \{\vec{v}_1, \ldots, \vec{v}_n\}$ basis for V_1
$B_2 = \{\vec{w}_1, \ldots, \vec{w}_n\}$ —— V_2

$$f \in \mathrm{Bil}_{\mathbb{K}}(V_1, V_2, \mathbb{K}) \rightsquigarrow Q \in \mathrm{Mat}_{n\times n}(\mathbb{K})$$

$$f(v,w) = [v]_{B_1}^T \, Q \, [w]_{B_2} \longleftrightarrow Q_{ij} = f(v_i, w_j)$$

Note: $[\varphi_1]_{B_1 B_2^*} = Q^T \qquad [\varphi_2]_{B_2 B_1^*} = Q$

Proposition: f is non-deg $\iff$ Q is invertible

Symmetric - Skew - symmetric & alternating forms

Def: We say f in $\mathrm{Bil}(V, V, \mathbb{K})$ is

1. symmetric if $f(v,w) = f(w,v)$ $\forall v,w \in V$
2. skew-symmetric if $f(v,w) = -f(w,v)$ $\forall v,w \in V$
3. alternating if $f(v,v) = 0$ $\forall v \in V$.

same unless char $\mathbb{K} = 2$

Lemma: $f \in \mathrm{Bil}(\mathbb{K}^n, \mathbb{K}^n, \mathbb{K})$ with associated matrix Q. Then

① f is symmetric if & only if $Q = Q^T$ (symmetric matrix)

② f is skew-symmetric if & only if $Q^T = -Q$ (skew-sym matrix)

③ f is alternating if & only if $Q^T = -Q$ & $Q_{ii} = 0$ $\forall i$.

Proposition: $\mathrm{Bil}(V, V, \mathbb{K}) \underset{\simeq}{\overset{\varphi}{}} \mathrm{Bil}^{\mathrm{sym}}(V, V, \mathbb{K}) \oplus \mathrm{Bil}^{\text{skew-sym}}(V, V, \mathbb{K})$

(for char $\mathbb{K} \neq 2$)

Lemma: If char $\mathbb{K} \neq 2$ $\dim V < \infty$, then $f \in \mathrm{Bil}^{\mathrm{sym}}(V, V, \mathbb{K})$ is completely determined by $f(v,v)$ $\forall v \in V$.

Q: How to work with degenerate symm. forms in $Bil(V, V, \mathbb{K})$?

Classification of real symmetric forms

GOAL Classify symmetric non-deg bilinear forms on $V \cong \mathbb{R}^n$ via invariants

STEP 1: Degenerate vs non-degenerate

STEP 2: Classify non-deg symm forms = Sylvester's Thm

Sylvester's Theorem: Fix $f : V \times V \longrightarrow \mathbb{R}$ non deg symm. bil form

Then $\exists$ basis $B = \{\varepsilon_1, \dots, \varepsilon_n\}$ of V s.t. $f(\varepsilon_i, \varepsilon_j) = \pm \delta_{ij}$

Moreover, the # of >0 is independent of the bases.

Part 1: Find $B = \{\varepsilon_1, \dots, \varepsilon_n\}$ basis for $\mathbb{R}^n$ with $h(\varepsilon_i, \varepsilon_j) = \pm\delta_{ij}$.

We have $\{w_1, \ldots w_n\}$ basis for $\mathbb{R}^n$ with $f(w_i, w_j) = 0 \ \forall i \neq j$