Lecture 39: Bilinear forms II (Symmetric forms)

Last Time: $\mathrm{Bil}_{\mathbb{K}}(V_1, V_2, \mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}(V_1 \otimes V_2, \mathbb{K}) = (V_1 \otimes V_2)^*$

- $f \in \mathrm{Bil}_{\mathbb{K}}(V_1, V_2, \mathbb{K})$ non deg $\iff$ induces linear injections

$$\begin{array}{ccc} V_1 \overset{\varphi_1}{\hookrightarrow} V_2^* & \qquad & V_2 \overset{\varphi_2}{\hookrightarrow} V_1^* \\ v \longmapsto f(v,-) & & v \longmapsto f(-,v) \end{array}$$

Prop: V_1, V_2 finite dimensional, f non-deg $\Rightarrow$ $\dim V_1 = \dim V_2$

$\overset{?}{\Leftarrow}$

A: If $V_1 \simeq \mathbb{K}^n \simeq V_2$ write $B_1 = \{\vec{v}_1, \dots, \vec{v}_n\}$ basis for V_1

$B_2 = \{\vec{w}_1, \dots, \vec{w}_n\}$ —— V_2

$f \in \mathrm{Bil}_{\mathbb{K}}(V_1, V_2, \mathbb{K}) \rightsquigarrow Q \in \mathrm{Mat}_{n\times n}(\mathbb{K})$

$$f(v,w) = [v]_{B_1}^T \, Q \, [w]_{B_2} \longleftrightarrow Q_{ij} = f(v_i, w_j)$$

Note: $[\varphi_1]_{B_1 B_2^*} = Q^T \qquad [\varphi_2]_{B_2 B_1^*} = Q$

Proposition: f is non-deg $\iff$ Q is invertible

Symmetric - Skew-symmetric & alternating forms

Def: We say f in $\mathrm{Bil}(V,V,\mathbb{K})$ is

1. **symmetric** if $f(v,w)=f(w,v)$ $\forall v,w\in V$
2. **skew-symmetric** if $f(v,w)=-f(w,v)$ $\forall v,w\in V$
3. **alternating** if $f(v,v)=0$ $\forall v\in V$.

(2) and (3): same unless $\mathrm{char}\,\mathbb{K}=2$

Ex 1 Dot Product on $V=\mathbb{R}^n$ is symmetric.

In general: $f(x,y)=x^TQy$ with $Q=Q^T$ gives a symmetric bilinear form on $\mathbb{K}^n$

Ex 2 $V=\mathbb{K}$ & $f(x,y)=xy$ is symmetric not alternating bilinear form. Skew symmetric when $\mathrm{char}\,\mathbb{K}=2$

Ex 3 $f((x,y),(x',y'))=xy'-x'y=\det\begin{pmatrix}x & x'\\ y & y'\end{pmatrix}$ is skew-symmetric & alternating bilinear form on $\mathbb{R}^2$

Ex ④ Pick $u \in \mathbb{R}^3$, $f_u(v,w) = u \cdot (v \times w)$ is alternating, skew-sym. form on $\mathbb{R}^3$.

Ex ⑤ $\dim V = n < \infty$ & $W = \mathrm{End}_{\mathbb{K}}(V) = \mathrm{Mat}_{n\times n}(\mathbb{K})$

$f \in \mathrm{Bil}(W, W, \mathbb{K})$ via $f(A, A') = \mathrm{Tr}(A A')$ Trace form

symmetric by construction.

Ex ⑥ $V = C[0,1] = \{h : [0,1] \longrightarrow \mathbb{R}$ cont$\}$. inf-dim'l $\mathbb{R}$-v.s.

- $F : V \times V \longrightarrow \mathbb{R}$ via $F(g,h) = \int_0^1 g(x)\, h(x)\, dx$

F is a symmetric bilinear form.

- Pick $k : [0,1]^2 \longrightarrow \mathbb{R}$ continuous:

$F_k : V \times V \longrightarrow \mathbb{R}$ via $F_k(g,h) = \iint_{[0,1]^2} g(x)\, h(y)\, k(x,y)\, dx$

- F_k bilinear but not symmetric. (unless k is).

Lemma: $f \in \mathrm{Bil}(\mathbb{K}^n, \mathbb{K}^n, \mathbb{K})$ with associated matrix Q. Then

1. f is symmetric if & only if $Q = Q^T$ (symmetric matrix)
2. f is skew-symmetric if & only if $Q^T = -Q$ (skew-sym matrix)
3. f is alternating if & only if $Q^T = -Q$ & $Q_{ii} = 0 \ \forall i$.

Proposition: $\mathrm{Bil}(V, V, \mathbb{K}) \overset{\varphi}{\simeq} \mathrm{Bil}^{\mathrm{sym}}(V, V, \mathbb{K}) \oplus \mathrm{Bil}^{\mathrm{skew\text{-}sym}}(V, V, \mathbb{K})$

(for char $\mathbb{K} \neq 2$)

Pf/ $\varphi(f) = f_1 + f_2$

$$f_1(v,w) = \frac{f(v,w) + f(w,v)}{2}$$

$$f_2(v,w) = \frac{f(v,w) - f(w,v)}{2}$$

□

Alternative: $(V \otimes V)^* \simeq V^* \otimes V^* \simeq S^2(V^*) \oplus \Lambda^2(V^*)$

Lemma: If char $\mathbb{K} \neq 2$, $\dim V < \infty$, then $f \in \mathrm{Bil}^{\mathrm{sym}}(V, V, \mathbb{K})$ is completely determined by $f(v,v)$ $\forall v \in V$.

Pf/ $f(v+w, v+w) = f(v, v+w) + f(w, v+w) = f(v,v) + f(v,w) + f(w,v) + f(w,w)$

$= f(v,v) + 2f(v,w) + f(w,w)$

So $f(v,w) = (f(v+w, v+w) - f(v,v) - f(w,w))/2$

□

Q: How to work with degenerate symm. forms in $\mathrm{Bil}(V,V,\mathbb{K})$?

Same idea works for skew-symmetric forms!

A: Given $f \in \mathrm{Bil}^{\text{symm}}(V,V,\mathbb{K})$, & $v \in V$, set

$$v^\perp = \{w \in V \mid f(v,w) = 0\}.$$

$$\mathrm{Rad}(f) = \{v \mid f(v,-) = 0 \in V^*\} \subseteq V \text{ subspace.}$$

If $V' = V/\mathrm{Rad}(f)$. View $V = W \oplus \mathrm{Rad}(f)$ (via a section) with $W \simeq V'$

Claim: $f|_{W\times W}$ is non-deg

Pf/ The matrix for f has the form

$$\begin{array}{ccc} & W & \mathrm{Rad}(f) & \\ & \left(\begin{array}{c|c} Q & 0 \\ \hline 0 & 0 \end{array}\right) & & \begin{array}{c} W \\ \mathrm{Rad}(f)\end{array} \end{array}$$

by symmetry

$\hookrightarrow f(v,-) = 0 \quad \forall v \in \mathrm{Rad}(f)$

If $w \in \ker(\varphi_1 : W \longrightarrow W^*)$
$\qquad\qquad w' \longmapsto f(w,w')$

then $f(w,w') = 0 \quad \forall w' \in W$.

But $f(w,v) = 0 \quad \forall v \in \mathrm{Rad}(f)$

$\Big\}$ $f(w,v) = 0 \ \forall v \in V$ so $w \in \mathrm{Rad}(f)$

Conclude $w \in W \cap \mathrm{Rad}(f) = 0$, so $\varphi_1 : W \hookrightarrow W^*$.

Since f is symmetric $\varphi_2 : W \hookrightarrow W^*$ as well. $\square$

Classification of real symmetric forms

GOAL Classify symmetric non-deg bilinear forms on $V \cong \mathbb{R}^n$ via invariants

STEP 1: Degenerate vs non-degenerate

① 1^{st} invariant = rank of f $= rk([f]) = \dim V - \dim Rad(f)$

STEP 2: Classify non-deg symm forms = Sylvester's Thm

Sylvester's Theorem: Fix $f : V \times V \longrightarrow \mathbb{R}$ non deg symmetric bil form

Then $\exists$ basis $B = \{\varepsilon_1, \dots, \varepsilon_n\}$ of V s.t. $f(\varepsilon_i, \varepsilon_j) = \pm \delta_{ij}$

Moreover, the # of >0 is independent of the bases.

Def Signature(f) = # 1's (For degenerate forms = (#1's, #−1's))

② 2^{nd} invariant = signature of f

- Proof involves 2 parts (1) Building basis $B \longleftrightarrow$ Gram-Schmidt algorithm

  (2) S is indep of $B \Rightarrow$ invariant of f

Part 1: Find $B = \{\varepsilon_1, \ldots, \varepsilon_n\}$ basis for $\mathbb{R}^n$ with $f(\varepsilon_i, \varepsilon_j) = \pm \delta_{ij}$

STEP 1 Gram-Schmidt algorithm: **INPUT**: $\{v_1, \ldots, v_n\}$ basis for $V \simeq \mathbb{R}^n$

OUTPUT: $\{w_1, \ldots, w_n\}$ basis for $V \simeq \mathbb{R}^n$ with $f(w_i, w_j) = 0$ if $i \neq j$

- $w_1 = v_1 \neq 0$ so $f(w_1, w_1) \neq 0$. ⚠
- Fix $1 \leq k < n$ & assume we've constructed $\{w_1, \ldots, w_k\}$ satisfying

  ① $Sp(v_1, \ldots, v_k) = Sp(w_1, \ldots w_k) \Rightarrow \{w_1, \ldots w_k\}$ li

  ② $w_i \neq 0$ & $f(w_i, w_j) = 0 \quad \forall i \neq j \quad 1 \leq i, j \leq k$ ($\Rightarrow f(w_i, w_i) \neq 0 \; \forall i$) ⚠

Then
$$w_{k+1} = v_{k+1} - \sum_{i=1}^{k} \frac{f(w_i, v_{k+1})}{f(w_i, w_i)} w_i$$

So
$$\begin{aligned} f(w_j, w_{k+1}) &= f\left(w_j, v_{k+1} - \sum_{i=1}^{k} \frac{f(w_i, v_{k+1})}{f(w_i, w_i)} w_i\right) \\ &= f(w_j, v_{k+1}) - \sum_{i=1}^{k} \frac{f(w_i, v_{k+1})}{f(w_i, w_i)} f(w_j, w_i) \\ &= f(w_j, v_{k+1}) - \frac{f(w_j, v_{k+1})}{f(w_i, w_i)} f(w_i, w_i) = 0 \end{aligned}$$

only $i = j$ survives

By construction: ① $Sp(v_1, \ldots v_{k+1}) = Sp(w_1, \ldots w_{k+1})$, so $\{w_1, \ldots w_{k+1}\}$ is li

② $w_i \neq 0 \; \forall i$ & $f(w_i, w_j) = 0 \quad \forall i \neq j \quad 1 \leq i, j \leq k$. □

We have $\{w_1, \ldots w_n\}$ basis for $\mathbb{R}^n$ with $f(w_i, w_j) = 0 \ \forall i \neq j$

STEP 2 : Set $s = \#\{i : f(w_i, w_i) > 0\}$. Then, reorder $\{w_1, \ldots, w_n\}$ so

$$\begin{cases} f(w_i, w_i) > 0 & \text{for } i \leq s \\ f(w_i, w_i) < 0 & \text{for } i > s \end{cases} \implies \text{Define } \boxed{\varepsilon_j} = \begin{cases} \dfrac{w_j}{\sqrt{f(w_j, w_j)}} & \text{for } j \leq s \\ \dfrac{w_j}{\sqrt{-f(w_j, w_j)}} & \text{for } j > s \end{cases}$$

($\neq 0$ because f is non-deg)

Then: $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is a bases for $\mathbb{R}^n$, $f(\varepsilon_i, \varepsilon_j) = 0 \ \forall i \neq j$ &

$$f(\varepsilon_j, \varepsilon_j) = \begin{cases} 1 & \text{for } j = 1, \ldots s \\ -1 & \text{for } j = s+1, \ldots, n \end{cases}$$

$\square$

Part 2: Show s is an invariant of f. (Next time!)

⚠ Step one fails if at any point $f(w_i, w_i) = 0$. Next time we will discuss an alternative proof that works.