Lecture 40: Bilinear forms III: Skew symmetric forms

Recall $f \in Bil(V,V,\mathbb{K})$ symmetric if $f(v,w) = f(w,v) \quad \forall v,w$

skew symm $f(v,w) = -f(w,v) \quad \forall v,w.$

GOAL: Classify (non-deg) symmetric / skew-symm forms for $\operatorname{char}\mathbb{K} \neq 2$

- Fix $V \simeq \mathbb{K}^n$ & $f \in Bil(V,V,\mathbb{K})$ symmetric or skew-symm

$$\boxed{Rad(f)} := \{ v \in V \mid f(v,-) = 0 \text{ in } V^* \}.$$

$\Rightarrow$ Can pick a basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ for V where

① $Sp(\mathcal{B}_2) = Rad(f)$, $W := Sp(\mathcal{B}_1)$ so $W \oplus Rad(f) = V$

② $\tilde{f} = f|_{W \times W} : W \times W \longrightarrow \mathbb{K}$ is non-degenerate

$$[f]_{\mathcal{B}\mathcal{B}} = \begin{array}{c c} & \begin{array}{cc} m & \end{array} \\ m & \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$$

Q in $GL_m(\mathbb{K})$.

$Rank(f) = rk\,Q$

- $Q = [\tilde{f}]_{\mathcal{B}_1\mathcal{B}_1}$ $\quad rk(f) = rk(\tilde{f}) = rk(Q)$

- $Q = Q^T$ if f is symmetric & $Q = -Q^T$ if f is skew-symmetric.

Symmetric bilinear $\mathbb{K}$-forms ($\mathrm{char}\,\mathbb{K} \neq 2$)

Recall: $f \in \mathrm{Bil}^{\mathrm{sym}}(V, V, \mathbb{K})$ $\mathrm{char}\,\mathbb{K} \neq 2$, then

$$f(v,w) = \frac{f(v+w, v+w) - f(v,v) - f(w,w)}{2} \quad \forall v, w \in V.$$

Theorem: Assume $\tilde{f} : \mathbb{K}^m \times \mathbb{K}^m \longrightarrow \mathbb{K}$ is non-deg bilinear form & $\mathrm{char}\,\mathbb{K} \neq 2$. Then, there exists $B = \{\varepsilon_1, \dots, \varepsilon_m\}$ basis for $\mathbb{K}^m$ with
$f(\varepsilon_i, \varepsilon_j) = 0$ $\forall i \neq j$ & $f(\varepsilon_i, \varepsilon_i) \neq 0$ $\forall i$.

If $\mathbb{K} = \overline{\mathbb{K}}$ we can further require $f(\varepsilon_i, \varepsilon_i) = 1$ $\forall i = 1, \dots, m$.

Lemma: Assume f is symm, non-deg $\mathbb{K}$-bilinear form, on $\mathbb{K}^n$, char $\mathbb{K} \neq 2$. Then

1. $\exists\, v_0 \neq 0$ with $f(v_0, v_0) \neq 0$
2. $\mathbb{K}^n = Sp(v_0) \oplus \langle v_0 \rangle^\perp$ where $\langle v_0 \rangle^\perp = \{ w \in \mathbb{K}^n : f(v, w) = 0 \}$
3. $f|_{\langle v_0 \rangle^\perp \times \langle v_0 \rangle^\perp}$ is a non-degenerate symmetric bilinear form.

③ <u>To show</u>: $F|_{\langle v_0\rangle^\perp \times \langle v_0\rangle^\perp}$ is a non-degenerate symmetric bilinear form.

<u>Proposition</u>: Assume $\tilde{F}: V \times V \longrightarrow \mathbb{K}$ is non-deg bilinear form, $\dim V = n$. Then, there exists $B = \{\varepsilon_1, \ldots, \varepsilon_n\}$ basis for V with.

$F(\varepsilon_i, \varepsilon_j) = 0 \quad \forall\, i \neq j$. & $F(\varepsilon_i, \varepsilon_i) \neq 0$. $\forall\, i$.

Theorem If $\overline{\mathbb{K}} = \mathbb{K}$ we can further require $h(\varepsilon_i, \varepsilon_i) = 1$ $\forall i$

Sylvester's Theorem: Fix $f: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ non deg symmetric bil form

Then $\exists$ basis $B = \{\varepsilon_1, \dots, \varepsilon_n\}$ of $\mathbb{R}^n$ s.t. $f(\varepsilon_i, \varepsilon_j) = \pm \delta_{ij}$

Moreover, the # of >0 is independent of the bases. (= signature of f)

Proof. By Proposition (slide 4), $\exists$ basis $B' = \{w_1, \dots w_n\}$ with $f(w_i, w_j) = 0 \quad \forall i \neq j$

$f(w_i, w_i) \neq 0 \quad \forall i$

$$\varepsilon_i = \frac{w_i}{\sqrt{|f(w_i, w_i)|}} \implies f(\varepsilon_i, \varepsilon_i) = \frac{f(w_i, w_i)}{|f(w_i, w_i)|} = \pm 1 \quad \forall i.$$

Set $\boxed{s} = \#\{i\text{'s} : f(\varepsilon_i, \varepsilon_i) = 1\}$

To finish: show s is independent of B

Say we have 2 basis as in the first part of the Theorem:

- $B = \{v_1, \dots, v_p, v_{p+1}, \dots v_n\}$ with $f(v_i, v_i) = \begin{cases} 1 & i \leq p \\ -1 & i > p \end{cases}$
- $B' = \{w_1, \dots, w_q, w_{q+1}, \dots, w_n\}$ — $f(w_i, w_i) = \begin{cases} 1 & i \leq q \\ -1 & i > q \end{cases}$
- $f(v_i, v_j) = f(w_i, w_j) = 0 \quad \forall i \neq j$

Consequence: Classification of quadratic forms q in $\mathbb{R}^n$

(= homogeneous degree 2 polynomials in $\mathbb{R}[x_1, \dots x_n]$)

After a linear change of coordinates, they become

$$x_1^2 + \dots + x_s^2 - x_{s+1}^2 - \dots - x_r^2$$

Skew-symmetric bilinear forms for char $\mathbb{K} \neq 2$

- Earlier discussion reduces to non-degenerate forms.

Lemma: If $\exists\, f \in \mathrm{Bil}^{\text{skewsym}}_{\text{nondeg}}(V, V, \mathbb{K})$ & $\dim V < \infty$ then $\dim V$ is even.

Theorem: Let $f : V \times V \longrightarrow \mathbb{K}$ be a non-deg skew-symmetric form on a finite-dimensional $\mathbb{K}$-vector space V, with char $\mathbb{K} \neq 2$.

Then $\exists$ basis $\mathcal{B} = \{\varepsilon_i, \eta_i\}_{i=1}^{n}$ for V s.t.

1. $f(\varepsilon_i, \eta_j) = \delta_{ij} = -f(\eta_j, \varepsilon_i) \quad \forall\, i, j$
2. $f(\varepsilon_i, \varepsilon_j) = 0 = f(\eta_i, \eta_j) \quad \forall\, i, j.$

<u>Proof</u>: $\dim V = 2n$. Need to find $B = \{\varepsilon_1, \eta_1, \ldots, \varepsilon_n, \eta_n\}$ with

$$[F]_{BB} = \begin{bmatrix} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & 0 \\ & \ddots & \\ 0 & & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} \end{bmatrix}$$