Lecture 40: Bilinear forms III: Skew symmetric forms

Recall $f \in Bil(V,V,\mathbb{K})$ symmetric if $f(v,w) = f(w,v)$ $\forall v,w$

skew symm $f(v,w) = -f(w,v)$ $\forall v,w$.

GOAL: Classify (non-deg) symmetric / skew-symm forms for $\operatorname{char}\mathbb{K} \neq 2$

- Fix $V \simeq \mathbb{K}^n$ & $f \in Bil(V,V,\mathbb{K})$ symmetric or skew-symm

$$\boxed{Rad(f)} := \{ v \in V \mid f(v, -) = 0 \text{ in } V^* \}.$$

$\Rightarrow$ Can pick a basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ for V where

(1) $Sp(\mathcal{B}_2) = Rad(f)$, $W := Sp(\mathcal{B}_1)$ so $W \oplus Rad(f) = V$

(2) $\tilde{f} = f|_{W\times W} : W \times W \longrightarrow \mathbb{K}$ is non-degenerate

$$[f]_{\mathcal{B}\mathcal{B}} = \begin{array}{c} \\ m \end{array}\!\!\begin{array}{c} m \\ \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$$

Q in $GL_m(\mathbb{K})$.

$Rank(f) = rk\, Q$

- $Q = [\tilde{f}]_{\mathcal{B}_1\mathcal{B}_1}$ $\quad rk(f) = rk(\tilde{f}) = rk(Q)$

- $Q = Q^T$ if f is symmetric & $Q = -Q^T$ if f is skew-symmetric.

$\boxed{\text{Symmetric bilinear } \mathbb{K}\text{-forms } (\operatorname{char}\mathbb{K} \neq 2)}$

Recall: $f \in \operatorname{Bil}^{\text{sym}}(V, V, \mathbb{K})$ $\operatorname{char}\mathbb{K} \neq 2$, then

$$f(v,w) = \frac{f(v+w, v+w) - f(v,v) - f(w,w)}{2} \qquad \forall\, v,w \in V.$$

Theorem: Assume $\tilde{f} : \mathbb{K}^m \times \mathbb{K}^m \longrightarrow \mathbb{K}$ is non-deg bilinear form & $\operatorname{char}\mathbb{K} \neq 2$. Then, there exists $B = \{\varepsilon_1, \dots, \varepsilon_m\}$ basis for $\mathbb{K}^m$ with $f(\varepsilon_i, \varepsilon_j) = 0$ $\forall\, i \neq j$ & $f(\varepsilon_i, \varepsilon_i) \neq 0$ $\forall\, i$.

If $\mathbb{K} = \overline{\mathbb{K}}$ we can further require $f(\varepsilon_i, \varepsilon_i) = 1$ $\forall\, i = 1, \dots, m$.

⚠ Tempting to use Gram-Schmidt to build B. **Issue** even if f is not degenerate, we cannot ensure $f(v,v) \neq 0$ if $v \neq 0$.

GS:
$$\begin{cases} w_1 = e_1 \\[2ex] w_{k+1} = e_{k+1} - \displaystyle\sum_{j=1}^{k} \frac{f(e_{k+1}, w_j)}{f(w_j, w_j)}\, w_j \qquad \text{for } k > 1 \end{cases}$$

This fails if at any point $f(w_j, w_j) = 0$. $\rightsquigarrow$ need a different proof strategy

<u>Lemma</u>: Assume f is symm, non-deg $\mathbb{K}$-bilinear form, on $\mathbb{K}^n$, char $\mathbb{K} \neq 2$. Then

① $\exists\, v_0 \neq 0$ with $f(v_0, v_0) \neq 0$

② $\mathbb{K}^n = \mathrm{Sp}(v_0) \oplus \langle v_0 \rangle^\perp$ where $\langle v_0 \rangle^\perp = \{ w \in \mathbb{K}^n : f(v,w) = 0 \}$

③ $f|_{\langle v_0 \rangle^\perp \times \langle v_0 \rangle^\perp}$ is a non-degenerate symmetric bilinear form.

Pf/ ① If $f(v,v) = 0 \;\; \forall v \in V \implies f(v,w) = 0 \;\; \forall v, w$ Contr!

② Build $W = \langle v_0 \rangle^\perp$. By ① $v_0 \notin W$.

Since $W = \ker \left(\begin{array}{ccc} \mathbb{K}^n & \xrightarrow{\varphi_{1(v_0)}} & \mathbb{K} \\ w & \longmapsto & f(v_0, w) \end{array} \right)$ we know it is a subspace of $\mathbb{K}^n$!

- <u>Claim</u>: $W \cap \mathrm{Sp}(v_0) = \{0\}$

Indeed, $f(v_0, \lambda v_0) = \lambda \underbrace{f(v_0, v_0)}_{\neq 0}$ so $\lambda v_0 \in W \iff \lambda = 0$.

- By the Rank-Nullity Theorem, $\dim W + \underbrace{\mathrm{rk}(\varphi_{1(v_0)})}_{=1 \text{ since } \varphi_1(v_0) \neq 0 \text{ in } (\mathbb{K}^n)^*} = n$ so $\dim W = n-1$.

- $\dim (W + \mathrm{Sp}(v_0)) \underset{\text{↳ Claim}}{=} n$ so $\mathbb{K}^n$

□

3 **To show:** $f|_{\langle v_0\rangle^\perp \times \langle v_0\rangle^\perp}$ is a non-degenerate symmetric bilinear form.

By symmetry, enough to check $\langle v_0\rangle^\perp = W \xrightarrow{\tilde{\varphi}} W^*$ is injective
$$w \longmapsto (v \longmapsto f(w,v))$$

- Pick $w \in \ker(\varphi)$ so $f(w,v) = 0 \quad \forall v \in W$

  But $w \in \langle v_0\rangle^\perp$ so $f(w, \lambda v_0) = \lambda f(w, v_0) = 0$.

  So $f(w, -) = 0 \in (\mathbb{K}^n)^*$ by 2.

This implies $w \in \ker(V \xrightarrow{\varphi} V^*) = \{0\}$. $\square$

Proposition: Assume $f: V \times V \longrightarrow \mathbb{K}$ is non-deg bilinear form, $\dim V = m$. Then, there exists $B = \{\varepsilon_1, \dots, \varepsilon_m\}$ basis for V with
$f(\varepsilon_i, \varepsilon_j) = 0 \quad \forall i \neq j$. & $f(\varepsilon_i, \varepsilon_i) \neq 0$. $\forall i$.

Pf/ Write $V \simeq \mathbb{K}^m$ & induct on m.

$\underline{m=1}$: nothing to check. Any $\varepsilon_1 \in \mathbb{K} \setminus \{0\}$ works because $V = \mathbb{K}$.
& f is non-deg.

<u>Induction step</u> : By Lemma, $\exists\, v_0 \in V \setminus \{0\}$ & $W = \langle v_0\rangle^{\perp}$ with $f(v_0, v_0) \neq 0$, $V = Sp(v_0) \oplus W$ & $f|_{W\times W}$ non-deg

- $\dim W = n-1$, so IH gives $\{\varepsilon_2, \dots, \varepsilon_{n-1}\}$ basis for W with
  $f(\varepsilon_i, \varepsilon_j) = 0 \quad \forall i \neq j$ & $f(\varepsilon_i, \varepsilon_i) \neq 0$.
- $\varepsilon_1 = v_0$ satisfies $f(\varepsilon_1, \varepsilon_i) = 0 \ \forall i > 1$ $(W \perp v_0)$
  $f(\varepsilon_1, \varepsilon_1) \neq 0$

$\square$

<u>Theorem</u> If $\overline{\mathbb{K}} = \mathbb{K}$ we can further require $f(\varepsilon_i, \varepsilon_i) = 1 \ \forall i$

Pf/ $a_i = f(\varepsilon_i, \varepsilon_i) \neq 0 \implies \exists\, \alpha \in \mathbb{K} \setminus \{0\}$ with $\alpha^2 = a_i$ (because $\mathbb{K} = \overline{\mathbb{K}}$). Normalize ε_i via $\dfrac{\varepsilon_i}{\alpha_i}$. Then:

- $f\left(\dfrac{\varepsilon_i}{\alpha_i}, \dfrac{\varepsilon_i}{\alpha_i}\right) = \dfrac{1}{\alpha_i^2} f(\varepsilon_i, \varepsilon_i) = \dfrac{a_i}{\alpha_i^2} = 1 \quad \forall i$
- $f\left(\dfrac{\varepsilon_i}{\alpha_i}, \dfrac{\varepsilon_j}{\alpha_j}\right) = \dfrac{1}{\alpha_i \alpha_j} f(\varepsilon_i, \varepsilon_j) = 0 \quad \forall i \neq j$

$\square$

Sylvester's Theorem: Fix $f: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ non deg symmetric bil form

Then $\exists$ basis $B = \{\varepsilon_1, \dots, \varepsilon_n\}$ of $\mathbb{R}^n$ s.t. $f(\varepsilon_i, \varepsilon_j) = \pm \delta_{ij}$

Moreover, the # of >0 is independent of the bases. (= signature of f)

Proof. By Proposition (slide 4), $\exists$ basis $B' = \{w_1, \dots w_n\}$ with $f(w_i, w_j) = 0 \quad \forall i \neq j$
$f(w_i, w_i) \neq 0 \quad \forall i$

$$\varepsilon_i = \frac{w_i}{\sqrt{|f(w_i, w_i)|}} \Rightarrow f(\varepsilon_i, \varepsilon_i) = \frac{f(w_i, w_i)}{|f(w_i, w_i)|} = \pm 1 \quad \forall i.$$

Set $\boxed{s} = \# \{i\text{'s} : f(\varepsilon_i, \varepsilon_i) = 1\}$

To finish: show s is independent of B

Say we have 2 basis as in the first part of the Theorem:

- $B = \{v_1, \dots, v_p, v_{p+1}, \dots v_n\}$ with $f(v_i, v_i) = \begin{cases} 1 & i \leq p \\ -1 & i > p \end{cases}$
- $B' = \{w_1, \dots, w_q, w_{q+1}, \dots, w_n\}$ — $f(w_i, w_i) = \begin{cases} 1 & i \leq q \\ -1 & i > q \end{cases}$
- $f(v_i, v_j) = f(w_i, w_j) = 0 \quad \forall i \neq j$

- Assume $p<q$ & reach a contradiction. ($\Rightarrow$ Symmetry yields $p=q$)

We define a linear transformation $L:\mathbb{R}^n \longrightarrow \mathbb{R}^{p+n-q}$ $\quad L(\xi) = \begin{bmatrix} f(v_1,\xi) \\ \vdots \\ f(v_p,\xi) \\ f(w_{q+1},\xi) \\ \vdots \\ f(w_n,\xi) \end{bmatrix}$

- $p+n-q<n$, so rank-nullity theorem gives $\xi_0 \in \operatorname{Ker} L \setminus \{0\}$.

- __Claim:__ $\xi_0 \in Sp(v_{p+1}, \dots v_n) \cap Sp(w_1, \dots, w_q)$

Pf/ $\xi_0 = \sum_{i=1}^{n} a_i v_i \Longrightarrow 0 = f(v_i,\xi_0) = a_i \underbrace{f(v_i,v_i)}_{\neq 0}\ \forall i \leq p$ so $\xi_0 \in Sp(v_{p+1} \dots v_n)$

$\xi_0 = \sum_{i=1}^{n} b_i w_i \Longrightarrow 0 = f(w_i,\xi_0) = -b_i\, f(w_i,w_i)\ \forall i > q$ so $\xi_0 \in Sp(w_1, \dots, w_q)$

Now $f(\xi_0,\xi_0) = f\Big(\sum_{i=p+1}^{n} a_i v_i, \sum_{i=p+1}^{n} a_i v_i\Big) = \sum_{i,j=p+1}^{n} a_i a_j\, f(v_i,v_j)$

$$= \sum_{i=p+1}^{n} -a_i^2 \underset{\xi_0 \neq 0}{<} 0$$

$$f(\xi_0,\xi_0) = f\Big(\sum_{i=1}^{q} b_i w_i, \sum_{i=1}^{q} b_i w_i\Big) = \sum_{i,j=1}^{q} b_i b_j\, f(w_i,w_j) = \sum_{i=1}^{q} b_i^2 \underset{\xi_0 \neq 0}{>} 0$$

$\rightsquigarrow$ Contradiction! $\square$

Consequence: Classification of quadratic forms q in $\mathbb{R}^n$
(= homogeneous degree 2 polynomials in $\mathbb{R}[x_1, \dots x_n]$)
After a linear change of coordinates, they become

$$x_1^2 + \dots + x_s^2 - x_{s+1}^2 - \dots - x_r^2$$

Pf/ • $f(v,w) = \frac{1}{2}\left(q(v_1+w_1, \dots, v_n+w_n) - q(\underline{v}) - q(\underline{w})\right) \in Bil(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R})$ (check!)

since $q = \sum_{i<j} a_{ij} x_i x_j + \sum_{i=1}^n a_i x_i^2$ & $q_{(\xi)} = f(\xi, \xi)$

• Set $r = \text{rank}(f)$

• Change of basis 1: $\{e_1, \dots, e_n\} \longrightarrow \{w_1, \dots, w_r, \underbrace{w_{r+1}, \dots, w_n}_{\text{in } Rad(f)}\}$

$\tilde{f} = f|_{\{w_1, \dots, w_r\}} : \mathbb{R}^r \times \mathbb{R}^r \longrightarrow \mathbb{R}$ symm & non deg, $s = \text{signature}(\tilde{f})$

By Sylvester's Theorem $\exists$ basis $\{\varepsilon_1, \dots, \varepsilon_r\}$ with • $f(\varepsilon_i, \varepsilon_i) = \begin{cases} 1 & i \leq s \\ -1 & i > s \end{cases}$
• $f(\varepsilon_i, \varepsilon_j) = 0$ if $i \neq j$

• Change of basis 2: $\{w_1 \dots w_r\} \longrightarrow \{\varepsilon_1 \dots \varepsilon_r\}$

Conclusion: $\{e_1, \dots, e_n\} \longleftrightarrow \{\varepsilon_1, \dots \varepsilon_r, w_{r+1}, \dots w_n\}$

$q \longmapsto \tilde{q}(\underline{x}) = x_1^2 + \dots + x_s^2 - x_{s+1}^2 - \dots - x_r^2.$

Skew-symmetric bilinear forms for char $\mathbb{K} \neq 2$

- Earlier discussion reduces to non-degenerate forms.

Lemma: If $\exists\ f \in \mathrm{Bil}^{\text{skew sym}}_{\text{non deg}}(V, V, \mathbb{K})$ & $\dim V < \infty$ then $\dim V$ is even.

Proof: Pick $Q = [f]_{\mathcal{B}\mathcal{B}} \in \mathrm{Mat}_{n \times n}(\mathbb{K})$ where $\mathcal{B}$ is any basis for V.

Then $Q^T = -Q$ since f is skew-symm. & Q is invertible (f non-deg)

Then $\det(Q) = \det(-Q^T) = (-1)^n \det Q^T = (-1)^n \det Q$

Since $\det Q \neq 0$ & char $\mathbb{K} \neq 2$ we have $(-1)^n = 1$ so n is even. $\square$

Theorem: Let $f : V \times V \longrightarrow \mathbb{K}$ be a non-deg skew-symmetric form on a finite-dimensional $\mathbb{K}$-vector space V, with char $\mathbb{K} \neq 2$.

Then $\exists$ basis $\mathcal{B} = \{\varepsilon_i, \eta_i\}_{i=1}^{n}$ for V s.t.

① $f(\varepsilon_i, \eta_j) = \delta_{ij} = -f(\eta_j, \varepsilon_i) \quad \forall\ i, j$

② $f(\varepsilon_i, \varepsilon_j) = 0 = f(\eta_i, \eta_j) \quad \forall\ i, j.$

$$[f]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}$$

(Alt: $[f]_{\varepsilon, \eta} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$)

Proof: $\dim V = 2n$. Need to find $B = \{\varepsilon_1, \eta_1, \ldots, \varepsilon_n, \eta_n\}$ with
$$[f]_{BB} = \begin{bmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$$

- Pick any $\varepsilon_1 \in V \setminus \{0\}$ & choose $\eta_1' \in V$ st $f(\varepsilon_1, \eta_1') \neq 0$

  (such η_1' exists since $f(\varepsilon_1, -) \neq 0$ in V^*)

  Claim: $\{\varepsilon_1, \eta_1'\}$ is li.

  Otherwise, $\eta_1' = \alpha \varepsilon_1$ (since $\varepsilon_1 \neq 0$), so $f(\varepsilon_1, \eta_1') = \alpha \underbrace{f(\varepsilon_1, \varepsilon_1)}_{=0} = 0$. Contd!

  Next, we rescale η_1' to $\eta_1 = \dfrac{\eta_1'}{f(\varepsilon_1, \eta_1')}$ so $f(\varepsilon_1, \eta_1) = 1$.

To finish, induct on n

- $n = 1$ $B = \{\varepsilon_1, \eta_1\}$ is the desired basis.
- $n > 1$. Pick $W = \langle \varepsilon_1, \eta_1 \rangle^{\perp} = \{w \in V : f(\varepsilon_1, w) = f(\eta_1, w) = 0\}$

Claim
- $W \cap Sp(\varepsilon_1, \eta_1) = \{0\}$ & $\dim W = n-2$ $\left(W = \operatorname{Ker}\left(V \twoheadrightarrow \mathbb{K}^2\right)\right)$ $\quad v \mapsto \begin{bmatrix} f(v, \varepsilon_1) \\ f(v, \eta_1) \end{bmatrix}$
- $f|_{W \times W}$ is non-deg (same idea as Lemma Slide 3)

  $W \oplus Sp(\varepsilon_1, \eta_1) = V$.

By IH $\exists B' = \{\varepsilon_2, \eta_2, \ldots, \varepsilon_{n-1}, \eta_{n-1}\}$ basis for W. with $[f|]_{B' \times B'} = \begin{bmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{bmatrix}$

$f(\varepsilon_1, \varepsilon_j) = f(\varepsilon_1, \eta_j) = 0 \quad \forall j > 1, \qquad f(\eta_1, \varepsilon_j) = f(\eta_1, \eta_j) = 0 \quad \forall j > 1.$ $\square$