ALGEBRA I (MATH 6111 AUTUMN 2021) - HOMEWORK 1

Problem 1. Let G and G' be two groups and $\varphi : G \to G'$ be a group homomorphism. Prove the following statements:

(i) φ is injective if and only if $\text{Ker}(\varphi) = \{e\}$.

(ii) φ is surjective if and only if $\text{Im}(\varphi) = G'$.

(iii) φ is an isomorphism if and only if it is a bijection.

Problem 2. Let $p \in \mathbb{Z}_{\geq 2}$ be a prime number. Let G be a group of order p. Prove that every non–identity element of G is a generator of G. (*Hence, G is cyclic, and in particular abelian*).

Problem 3. Let G be a group and H_1, H_2 be two subgroups of G. Assume that $G = H_1 \cup H_2$. Prove that either $G = H_1$ or $G = H_2$.

Problem 4. Let G be a group and H_1, H_2 be two subgroups of G such that both $(G : H_1)$ and $(G : H_2)$ are finite. Prove that $(G : H_1 \cap H_2)$ is also finite.

Problem 5. Consider the set $\mathbb{Q}$ of rational numbers viewed as an abelian group under usual addition.

(i) Is $\mathbb{Q}$ a finitely–generated group?
(ii) Does there exist a proper subgroup $H < \mathbb{Q}$ of finite index?

Problem 6. Let G be a group and H be a subgroup of G with $(G : H) = 2$. Prove that H is a normal subgroup of G.

Problem 7. Let G be a group such that every non–identity element of G has order 2 (so the exponent of G is 2). Prove that G is abelian.

Problem 8. Let G be a group of order ≤ 5. Prove that G is abelian. Give an example of a non–abelian group of order 6.

Problem 9. Let m, n be two positive integers. What is the cardinality of the set of group homomorphisms $\text{Hom}_{\text{Gps}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$?

Problem 10. Let n be a positive integer. Determine the cardinality of the set of group automorphisms $\text{Aut}_{\text{Gps}}(\mathbb{Z}/n\mathbb{Z})$.

Problem 11. Let G be a group. Consider the following subset of G:

$$X = \left\{ [a, b] := aba^{-1}b^{-1} \mid a, b \in G \right\} \subset G \ .$$

Let $H = \langle X \rangle$ be the subgroup of G generated by X. It is usually denoted by $[G : G]$ and it is known as the *commutator subgroup* of G. Prove the following assertions about H:

(i) H is a normal subgroup of G.

(ii) G/H is abelian.

(iii) If G' is an abelian group and $\psi : G \to G'$ is a group homomorphism, then $H \subset \mathrm{Ker}(\psi)$.

Problem 12. Let G be a group. Given $g \in G$, consider the (conjugation) map $C_g : G \to G$

$$C_g(x) = gxg^{-1} \text{ for every } x \in G.$$

(1) Prove that C_g is an automorphism of G.

(2) Prove that $C : G \to \mathrm{Aut}_{\mathrm{Gps}}(G)$ defined by $g \mapsto C_g$ is a group homomorphism.

(3) Prove that $\mathrm{Im}(C) \subset \mathrm{Aut}_{\mathrm{Gps}}(G)$ is a normal subgroup (called the group of inner automorphisms of G).

Problem 13. Let G be a finite abelian group (written additively), and let $H < G$ be

$$H = \{g \in G : 2g = 0\}.$$

Let $x \in G$ be defined as $x = \sum_{g \in G} g$. Prove that

(i) $x = \sum_{h \in H} h$.

(ii) If $|H| \neq 2$, then $x = 0$.

(iii) If $|H| = 2$, then $H = \{0, x\}$.

Problem 14. Let $p \in \mathbb{Z}_{\geq 2}$ be a prime number. Consider the group $G = (\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}$ under multiplication. Use the previous problem to show that $(p-1)! \equiv -1$ modulo p.

Problem 15. Let G be the group of symmetries of a regular hexagon. What is the order of G?

Problem 16. Let G be a finite group and N_1, N_2 be two normal subgroups of G. Assume that $|N_1|$ and $|N_2|$ are coprime.

(i) Prove that $x_1 x_2 = x_2 x_1$ for every $x_1 \in N_1$ and $x_2 \in N_2$.

(ii) Prove that $N_1 \cap N_2 = \{e\}$.

Problem 17. Let G be a group and N_1, N_2 be two normal subgroups of G. Assume that $N_1 \cap N_2 = \{e\}$. Prove that $x_1x_2 = x_2x_1$ for every $x_1 \in N_1$ and $x_2 \in N_2$.

Problem 18. Let G be a group. The *center* of G, denoted by $\mathsf{Z}(G)$, is defined as:

$$\mathsf{Z}(G) = \{g \in G : gx = xg \text{ for every } x \in G\}.$$

(i) Prove that $\mathsf{Z}(G)$ is a normal subgroup of G.

(ii) Assume that there is a subgroup $H < \mathsf{Z}(G)$ such that G/H is cyclic. Prove that G is abelian.

Problem 19. The group of quaternions Q_8 is defined as the set $\mathsf{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ subject to the following relations:

$$i^2 = j^2 = k^2 = -1,\ (-1)^2 = 1, ij = k, jk = i, ki = j, (-1)x = x(-1) = -x \text{ for } x = \pm i, \pm j, \pm k$$

This operation is associative (you do not have to prove this).

(i) Prove that every subgroup of Q_8 is normal.

(ii) Let D_4 be the dihedral group of order 8. It is the group of symmetries of a square, or more explicitly:

$$D_4 = \{e, \rho, \rho^2, \rho^3, s, s\rho, s\rho^2, s\rho^3\}$$

with group operation determined by: $s^2 = \rho^4 = e$ and $s\rho s = \rho^3$. Show that Q_8 and D_4 are not isomorphic.

Problem 20. Consider the Heisenberg group over $\mathbb{Z}/3\mathbb{Z}$ (viewed as the field with three elements):

$$\mathsf{H} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}/3\mathbb{Z} \right\}$$

with the group operation being matrix multiplication. Prove that $\exp(\mathsf{H}) = 3$, and that H is not abelian.