ALGEBRA I (MATH 6111 AUTUMN 2021) - HOMEWORK 2

Notations: S_n is the group of permutations on n letters $\{1, \ldots, n\}$. For $1 \leq i \leq n-1$, $s_i = (i\ i+1)$ denotes the simple transposition exchanging i and $i+1$.

Problem 1. Let G be a group acting on a set X. Assume that the action is free and transitive. Pick $x \in X$ and define a set map $G \to X$ by $g \mapsto g \cdot x$. Prove that this map is bijective.

Problem 2. Consider the following group acting on $X = \mathbb{R}^2 \smallsetminus \{(0,0)\}$:

$$G := \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : 0 \leq \theta < 2\pi \right\}$$

Describe the orbits of G.

Problem 3. Let G be a group and H be a subgroup of G. Consider the action of G on the left cosets G/H by $x \cdot (gH) = (xg)H$.

(i) Show that this is indeed an action of G on G/H.

(ii) What is the stabilizer of a left coset $gH \in G/H$?

(iii) Prove that this action is faithful (that is the group homomorphism $G \to \mathrm{Aut}_{\mathrm{Set}}(X)$ is injective) if, and only if, $\bigcap_{g \in G} gHg^{-1} = \{e\}$.

Problem 4. Assume G is a group and H is a subgroup of finite index, i.e., $(G : H) < \infty$. Prove that there exists a normal subgroup N of G such that $(G : N) < \infty$ with $N \subseteq H$. (*Hint:* Consider G acting on the finite set G/H.)

Problem 5. Let $G = \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$ (invertible 2×2 matrices over the field with three elements), and view G acting on itself by conjugation, that is $g \cdot h = ghg^{-1}$. Consider the following element of G:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Describe the orbit of X and its stabilizer subgroup.

Problem 6. (Projective linear group and the moduli space $M_{0,n}$) Let $\mathbb{K}$ be any field. We define the $(n-1)$-dimensional projective space over $\mathbb{K}$ as the set

$$\mathbb{P}^{n-1} := \{\underline{x} := (x_1, \ldots, x_n) \in \mathbb{K}^n \smallsetminus \{(0, \ldots, 0)\}\} / \sim,$$

where $\underline{x} \sim \underline{y}$ if, and only if, there exists $\lambda \in \mathbb{K} \smallsetminus \{0\}$ with $x_i = \lambda y_i$ for all $i = 1, \ldots, n$. We represent the class of a point $\underline{x}$ in $\mathbb{P}^{n-1}$ by $(x_1 : \ldots : x_n)$.

The *projective linear group* is defined as the quotient group:

$$\mathrm{PGL}(n) = \mathrm{GL}(n)/\mathsf{Z}(\mathrm{GL}(n)).$$

(i) Show that the center $\mathsf{Z}(\mathrm{GL}(n))$ equals the set of scalar matrices, i.e. the diagonal matrices $\mathrm{diag}(\lambda, \dots, \lambda)$ with $\lambda \in \mathbb{K} \smallsetminus \{0\}$. (*Hint:* Use permutation matrices to show that each matrix in the center is parameterized by two values: one for the diagonal entries and one for the off-diagonal entries. To finish, use elementary matrices to show that the off-diagonal entries must all be zero.)

(ii) Show that $\mathrm{PGL}(n)$ acts on $\mathbb{P}^{n-1}$ by left matrix multiplication.

(iii) Set $n = 2$ and consider the action of $\mathrm{PGL}(2)$ on sets of three distinct points in $\mathbb{P}^1$ (ordered triples of distinct points in $\mathbb{P}^1$) via

$$\sigma \cdot \{p_1, p_2, p_3\} = \{\sigma \cdot p_1, \sigma \cdot p_2, \sigma \cdot p_3\}.$$

Show that the action is transitive. Thus, we can represent the unique orbit by the set $\{0, 1, \infty\}$, that is $\{(0 : 1), (1 : 1), (1 : 0)\}$. (In geometric terms, this says that the moduli space of rational curves with three distinct marked points (denoted by $M^\circ_{0,3}$) is just a point).

(iv) Show that the $\mathrm{PGL}(2)$-orbits of tuples of four distinct ordered points in $\mathbb{P}^1$ is in bijection with $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$. (In geometric terms, this says $M^\circ_{0,4} = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$).

Problem 7. Consider the presentation of the dihedral group $D_n := \langle s, \rho | s^2, \rho^n, (s\rho)^2 \rangle$.

(i) Show that D_n admits an alternative presentation $D_n = \langle s_1, s_2 | s_1^2, s_2^2, (s_1 s_2)^n \rangle$.

(ii) Show that there exists a unique group homomorphism $f \colon D_n \to \{\pm 1\}$ with $f(s) = -1$ and $f(\rho) = 1$. Describe the kernel of f.

(iii) Consider the group homomorphism $D_n \hookrightarrow \mathrm{GL}_2(\mathbb{R})$ sending

$$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \rho \mapsto \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

where $\theta = \frac{2\pi}{n}$. Consider the action of D_n on the set $X := \mathbb{R}^2 \smallsetminus \{\mathbf{0}\}$ induced by the action of $\mathrm{GL}_2(\mathbb{R})$ on X. Determine all orbit sizes and characterize which subgroups of D_n are stabilizers.

Problem 8. Given a permutation $\pi \in S_n$, we define its *length* $\ell(\pi)$ as the smallest number ℓ such that π can be written as a product of ℓ simple transpositions. Prove that $\ell(\pi s_k) < \ell(\pi)$ if, and only if $\pi(k) > \pi(k+1)$.

Problem 9. Fix a permutation $\pi \in S_n$. Prove that $\ell(\pi)$ is the same as the cardinality of the following set

$$\{(i, j) : 1 \leq i < j \leq n \text{ and } \pi(i) > \pi(j)\}.$$

Problem 10. Let G_n be the group given by the following presentation. The set G_n has $n-1$ generators $g_1, \ldots, g_{n-1}$ and these generators satisfy the following list of relations:

$$\begin{aligned} g_i^2 &= e \quad \text{for every } 1 \leq i \leq n-1, \\ g_i g_j &= g_j g_i \quad \text{for every } 1 \leq i, j \leq n-1 \text{ with } |i-j| > 1, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \quad \text{for every } 1 \leq i \leq n-2. \end{aligned}$$

(i) Prove that there is a unique surjective group homomorphism $G_n \to S_n$ sending g_i to s_i for all $i = 1, \ldots, n-1$.

(ii) Let H be the subgroup of G_n generated by $g_1, \ldots, g_{n-2}$. Prove that the following is the list of all cosets G_n/H:

$$H_0 := H;\ H_1 := g_{n-1}H;\ H_2 := g_{n-2}H_1 = g_{n-2}g_{n-1}H; \ldots; H_{n-1} := g_1 H_{n-2} = g_1 \cdots g_{n-1} H.$$

(iii) Prove by induction on n that $|G_n| \leq n!$. Conclude that $G_n \xrightarrow{\sim} S_n$.

Problem 11. Determine the conjugacy classes in S_5 and the number of elements in each class.

Problem (Bonus). Show that the number of conjugacy classes in S_n is counted by the *partitions* of n. These are defined as non-increasing sequences $\lambda_1 \geq \lambda_2 \geq \ldots$ of non-negative integers with $\lambda_1 + \lambda_2 + \ldots = n$. Compute the number of elements in each conjugacy class.

Partitions are fundamental objects in enumerative combinatorics, and are usually denoted by $\lambda \vdash n$.) Do some literature search (e.g. using *Google*) to see how to count these partitions in terms of n and write a brief summary.