ALGEBRA I (MATH 6111 AUTUMN 2020) - HOMEWORK 3

Assumption: All groups considered in this problem set are finite.

Problem 1. Assume a finite group G acts *transitively* on a finite set X with $|X| \geq 2$. Prove that there exists $g \in G$ such that $X^g = \emptyset$. Conclude that the projection to the first component of the incidence variety $F = \{(g, x) \in G \times X : g \cdot x = x\}$ is not surjective.

Problem 2. Let G be a group and $H < G$ be a subroup. Let $P < H$ be a Sylow p-subgroup of H. Prove that there exists a Sylow p-subgroup of G (call it Q), satisfying $P = Q \cap H$.

Problem 3. Let G be a group and $N \triangleleft G$ be a normal subgroup. Consider the natural surjection $\pi \colon G \to G/N$.

(i) Let P be a Sylow p-subgroup of G. Prove that $P \cap N$ is a Sylow p-subgroup of N.

(ii) Prove that $\pi(P)$ is a Sylow p-subgroup of G/N.

Problem 4. Let $P < G$ be a Sylow p-subgroup of G, and let $N = N_G(P)$ be the normalizer of P (that is, $N = \{g \in G \colon gPg^{-1} = P\}$). Prove that for every $L < G$ containing N, $N_G(L) = L$.

Problem 5. Let $H \triangleleft G$ be a normal subgroup of a group G. Let us assume that $|H| = p$. Prove that H is contained in every Sylow p-subgroup of G.

Problem 6. Let $K \triangleleft G$ be a normal subgroup of a group G, and let P be a Sylow p-subgroup of K. Prove that $G = KN_G(P) := \{kh \colon k \in K, h \in N_G(P)\}$.

Problem 7. Let $G = GL_2(\mathbb{Z}/5\mathbb{Z})$.

(i) Show that $|G| = 480$ (Hint, count the number of vectors that you can place on each column of a matrix in G).

(ii) Show that $P := \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z}/5\mathbb{Z}\}$ and $Q := \{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x \in \mathbb{Z}/5\mathbb{Z}\}$ are 5-Sylow subgroups of G.

(iii) Use Sylow Theorem (B2) to conclude that $n_5 = 6$.

Problem 8. Let $G = GL_n(\mathbb{Z}/p\mathbb{Z})$ where $n \in \mathbb{Z}_{\geq 2}$.

(i) Show that $|G| = p^r m$ with $(m : p) = 1$ and $r = n(n-1)/2$.

(ii) Show that the set H of upper triangular matrices with 1's along the diagonal is a Sylow p-subgroup of G.

(iii) Show that the normalizer of H is the set of all invertible upper triangular matrices. (*Hint:* Use convenient elementary matrices in H to show any $A \in G$ with $AHA^{-1} = H$ must be upper triangular.)

(iv) Conclude from this that the number n_p of Sylow p-groups of G equals:

$$n_p = \prod_{k=1}^{n} (p^{k-1} + p^{k-2} + \ldots + 1) =: [n!]_p.$$

Problem 9. Assume that G is a non-abelian group of order p^3.

(i) Prove that $\mathsf{Z}(G) \simeq \mathbb{Z}/p\mathbb{Z}$.

(ii) Prove that $G/\mathsf{Z}(G) \simeq (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$.

(iii) If $H < G$ is a subgroup of G of order p^2, show that H is normal and contains $\mathsf{Z}(G)$.

Problem 10. Let p, q be two prime numbers with $p < q$. Fix a group G of order pq.

(i) Prove that G is not simple.

(ii) Further assume that $q \not\equiv 1 \pmod{p}$. Prove that G is cyclic.

Problem 11. Prove that there is no simple group of order p^2q, where p, q are prime numbers (not necessarily distinct).

Problem 12. Let $a \in \{1, 2, \ldots, p-1\}$ and $k \in \mathbb{Z}_{\geq 1}$. Prove that there is no simple group of order $p^k a$.

Problem 13. For each of the following numbers, prove that there is no simple group of that order: (a) 12; (b) 40; (c) 216.

Problem 14. Describe a Sylow 2-subgroup of the Dihedral group D_{10} of order 20.

Problem 15. Let G be a simple group of order 60 (assume it exists!). How many Sylow p-subgroups are there in G, for $p = 2, 3$ and 5?

Problem 16. (Key idea in Sylow's original proof of Thm (A))
Consider $H < G$ two groups, and fix a prime $p > 0$ with $p \,|\, |H|$. Assume G has a Sylow p-group, called P. We aim to prove that the same is true for H. We write $|H| = p^s m$ with $(m : p) = 1$ and $|G| = p^r m'$ with $(m' : p) = 1$ and $s \leq r$.

(i) Consider the left cosets $X := G/P$. Show that left multiplication defines an action of G on X.

(ii) Consider the action of H on X inherited from the action on item (i). Show that there is a left coset gP whose H-orbit has size coprime to p.

(iii) Show that p^s divides $|\operatorname{Stab}_H(gP)|$.

(iv) Show that $\operatorname{Stab}_G(gP) = gPg^{-1}$. Conclude that $|\operatorname{Stab}_G(gP)| = |P|$ and, therefore, $\operatorname{Stab}_G(gP)$ is a Sylow p-subgroup of G.

(v) Conclude from (iii) and (iv) that $|\operatorname{Stab}_H(gP)| = p^s$, so it is a Sylow p-subgroup of H.

Sylow's original proof realizes G as a subgroup of S_n for $n = |G|$ via the action of G on itself by left multiplication. In turn, given a prime p with $p|n$, S_n is viewed as a subgroup of $\mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$ sending a permutation σ to the permutation nmatrix P_σ, where

$$(P_\sigma)_{ij} = \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

By Problem 8, $\mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$ has a p-Sylow subgroup. Then, item (v) shows that the same is true for G.

Problem (Bonus). The objective of this problem is to prove the following stronger version of Sylow's Theorem (A).

Theorem. Given G of order $n := p^r m$ with $(m : p) = 1$, we can find subgroups $H_i < G$ with $|H_i| = p^i$ for all $i = 0, 1, \ldots, r$.

To prove the statement we will show that given $i < r$ and a subgroup H with $|H| = p^i$ we can find a p-subgroup H' of G with $H \subset H'$ and $(H' : H) = p$. Then, H' will satisfy $|H'| = p^{i+1}$.

(i) Show that the normalizer $N_G(H)$ satisfies $H < N_G(H)$ and $(N_G(H) : H) \equiv (G : H) \pmod p$ (*Hint:* Use the action of H on $X := G/H$ by left multiplication.)

(ii) Show the group $N_G(H)/H$ satisfies $p \,|\, |N_G(H)/H|$.

(iii) Assuming Sylow Theorem (A), show that $N_G(H)/H$ as an element σH of order p.

(iv) Conclude that $H' = \langle \sigma, H \rangle$ has the desired properties.

Note: item (iii) can be proved without assuming Sylow Theorem (A). This result is known as *Cauchy's Theorem*: Every finite group G with $p \,|\, |G|$ contains an element of order p. It can be proven for abelian groups by induction on the order of the group. In the general case, the result follows by working with either $Z(G)$ (if $p||Z(G)|$) or a suitable centralizer

$$C_G(g) := \{h \in G : hgh^{-1} = g\}$$

where $g \notin Z(G)$ and p does not divide the size of the conjugacy class of g.