ALGEBRA I (MATH 6111 AUTUMN 2021) - HOMEWORK 4

Problem 1. Fix a prime number $p > 2$. Assume G is a group of order $2p$. Prove that either G is cyclic or G is isomorphic to the Dihedral group D_p.

Problem 2. Let $C \colon G \to \mathrm{Aut}_{gp}(G)$ be given by $g \mapsto C_g$, where $C_g(x) = gxg^{-1}$ (see Problem 12 of Homework 1). Is $G \rtimes_C G$ isomorphic to $G \times G$?

Problem 3. Let A, B be two groups and $G = A \times B$. Let H be a subgroup of G such that $A \subseteq H$. Prove that $H = A \times (H \cap B)$.

Problem 4. Assume that there is a short exact sequence of group homomorphisms:

$$\mathbf{1} \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} \mathbb{Z} \longrightarrow \mathbf{1}\ .$$

Further assume that $\mathrm{Im}(\varphi) \subseteq \mathsf{Z}(B)$ (the center of B). Prove that this exact sequence is trivial (in particular, $B = A \times \mathbb{Z}$).

Problem 5. Let G_1 and G_2 be two groups and G be a subgroup of $G_1 \times G_2$. Let $\pi_1 \colon G_1 \times G_2 \to G_1$ and $\pi_2 \colon G_1 \times G_2 \to G_2$ be the two natural projections. Define

$$N_1 := G \cap G_1; \quad H_1 := \pi_1(G); \quad N_2 := G \cap G_2; \quad H_2 := \pi_2(G).$$

Prove that N_1 is normal in H_1 and N_2 is normal in H_2. Prove that there exists an isomorphism $H_1/N_1 \to H_2/N_2$.

Problem 6. Recall that S_n is the group of permutations of $\{1, \ldots, n\}$ and the length $\ell(\sigma)$ of a permutation σ in S_n was defined in Problem 8 of Homework 2. Show that it induces a natural group homomorphism $S_n \to \{\pm 1\}$ via $\sigma \mapsto (-1)^{\ell(\sigma)}$. We call it the *sign homomorphism* and define the alternating group A_n as the kernel of this homomorphism.

Problem 7. For each of the following short exact sequences, determine whether it is split and/or trivial. In each case, write a section and/or a retraction. Are sections/retractions unique?

(i) Recall that $\mathrm{SL}_2(\mathbb{C})$ is the group of 2×2 matrices of determinant 1, and $\mathrm{GL}_2(\mathbb{C})$ is the group of invertible 2×2 matrices (with entries from the field of complex numbers). The following is the short exact sequence associated to the determinant map $\det \colon \mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}^*$, where $\mathbb{C}^* := \mathbb{C} \smallsetminus \{0\}$:

$$\mathbf{1} \longrightarrow \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_2(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^* \longrightarrow \mathbf{1}.$$

(ii) Consider the natural inclusion of $\mathbb{Z}/2\mathbb{Z}$ in$\mathbb{Z}/4\mathbb{Z}$ and the following short exact sequence arising from it

$$\mathbf{0} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbf{0}.$$

(iii) Let sign: $S_3 \to \{\pm 1\}$ be the sign homomorphism from Problem 6 and consider the short exact sequence arising from it:

$$\mathbf{1} \longrightarrow A_3 \longrightarrow S_3 \xrightarrow{\text{sign}} \{\pm 1\} \longrightarrow \mathbf{1}.$$

Problem 8. Show that dihedral group D_4 and the quaternion group Q_8 are *extensions* of $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$, that is, we can fit both groups D_4 and Q_8 as G into a short exact sequences of the form:

$$\mathbf{1} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbf{1}.$$

In addition, show that these two groups are non-isomorphic.

Problem 9. The following examples highlight the differences between short exact sequences, split s.e.s. and trivial s.e.s.

(i) Show that the short exact sequence for Q_8 defined in Problem 8 is not split (*Hint:* Look at the elements of order 2 in Q_8).

(ii) Fix $n \geq 3$. Show that the short exact sequence

$$\mathbf{1} \longrightarrow A_n \longrightarrow S_n \xrightarrow{\text{sign}} \{\pm 1\} \longrightarrow \mathbf{1}$$

induced by the sign homomorphism from Problem 6 is split but not trivial. Conclude that $S_n \simeq A_n \rtimes \mathbb{Z}_2$ but $S_n \not\simeq A_n \times \mathbb{Z}_2$ (i.e., the direct product of A_n and $\mathbb{Z}_2$).

Problem 10. Find all groups (up to isomorphism) which will fit in the following short exact sequence:

$$\mathbf{0} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbf{0}.$$

Problem 11. Consider the following set of elements in $A_4 \subset S_4$:

$$\{id, (12)(34), (13)(24), (14)(23)\}.$$

(i) Prove that they form a normal subgroup in S_4 isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(ii) Prove that the following short exact sequence splits:

$$\mathbf{1} \longrightarrow (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow A_4 \longrightarrow A_3 \longrightarrow \mathbf{1}.$$

(iii) Decide if the following short exact sequence splits:

$$\mathbf{1} \longrightarrow (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow S_4 \longrightarrow S_3 \longrightarrow \mathbf{1}.$$

Problem 12. Let A, B, C be three **abelian groups** and let there be a short exact sequence

$$\mathbf{0} \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \mathbf{0}.$$

Prove that if this sequence splits, then it is trivial.

Problem 13. How many (up to isomorphism) groups of order 18 are there?

Problem 14. Compute the following automorphism groups: (a) $\mathrm{Aut}_{gp}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$, (b) $\mathrm{Aut}_{gp}(\mathbb{Z}/48\mathbb{Z})$.

Problem 15. Decide if $\mathrm{Aut}_{gp}(S_5) \simeq S_5$ or not.

Problem 16. Let B and $N \leq B$ be the following groups of matrix in $\mathrm{GL}_2(\mathbb{C})$.

$$B = \left\{ \begin{pmatrix} d_1 & x \\ 0 & d_2 \end{pmatrix} \text{ where } d_1, d_2 \neq 0 \text{ and } x \text{ is arbitrary} \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ where } x \text{ is arbitrary} \right\}$$

Prove that N is normal in B (we know this to be true for matrices over $\mathbb{Z}/p\mathbb{Z}$ by Problem 8 (iii) of Homework 3). Prove that B is a semidirect product of N and H, where H is the subgroup of B consisting of diagonal matrices.

Problem 17. Let H and N be two groups and $\alpha, \beta : H \to \mathrm{Aut}_{gp}(N)$ two group homomorphisms. Consider the semidirect products $A = N \rtimes_\alpha H$, and $B = N \rtimes_\beta H$.

(1) Assume that there exists $T \in \mathrm{Aut}_{gp}(N)$ such that $\beta(h)(n) = T(\alpha(h)(T^{-1}(n)))$ for every $h \in H$ and $n \in N$. Write an isomorphism $F_T : A \to B$.

(2) Assume that there exists $\psi \in \mathrm{Aut}_{gp}(H)$ such that $\alpha(h) = \beta(\psi(h))$, for every $h \in H$. Write an isomorphism $F_\psi : A \to B$.

(3) Assume that there exists a group homomorphism $j : H \to N$ such that

$$\alpha(h)(n) = j(h) \cdot (\beta(h)(n)) \cdot j(h)^{-1},$$

for every $h \in H$ and $n \in N$, where $\beta(h)(j(h')) = j(h')$ for all $h, h' \in H$. Write an isomorphism $F_j : A \to B$.