ALGEBRA I (MATH 6111 AUTUMN 2021) - HOMEWORK 5

Problem 1. (Fun with commutators)

Let G be a group. For $a, b \in G$, define $[a, b] := aba^{-1}b^{-1}$. Recall that for any two subsets $A, B \subset G$, we defined $[A, B]$ to be the subgroup generated by $\{[a, b] \colon a \in A, b \in B\}$.

(i) Verify the following identity, for all $a, x, y \in G$:

$$[a, xy] = [a, x][x, [a, y]][a, y].$$

(ii) Let A, B, C be three normal subgroups of G. Prove that $[A, [B, C]]$ is generated by $\{[a, [b, c]] \colon a \in A, b \in B, c \in C\}$.

(iii) Recall that $C^1(G) = G$ and $C^{n+1}(G) := [G, C^n(G)]$ defines the lower central series of G. Prove that for every $m, n \geq 1$ we have $[C^m(G), C^n(G)] \subseteq C^{m+n}(G)$.

Problem 2. Consider the following groups of matrices over $\mathbb{C}$.

$$B = \left\{ \begin{pmatrix} d_1 & x \\ 0 & d_2 \end{pmatrix} \text{ where } d_1, d_2 \neq 0 \text{ and } x \text{ is arbitrary} \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ where } x \text{ is arbitrary} \right\}$$

(i) Show that B is solvable.

(ii) Show that N and B/N are nilpotent, but B is not.

Problem 3. Let G be a group and let N_1, N_2 be two normal subgroups satisfying $[G, N_1] \subseteq N_2 \subseteq N_1$. Given any subgroup $H < G$, prove that $N_2H \triangleleft N_1H$.

Problem 4. Let $\varphi \colon G \to G'$ be a group homomorphism. Assume Σ' is a composition series of G':

$$\Sigma' \colon G' = G'_0 \triangleright G'_1 \triangleright G'_2 \triangleright \ldots \triangleright G'_n = \{e\}.$$

Let Σ be the sequence with terms $G_j = \varphi^{-1}(G'_j)$ for all $j = 0, \ldots, n$, and $G_{n+1} = \{e\}$.

(i) Prove that Σ is a composition series of G.

(ii) Prove that we have injective homomorphisms $\mathrm{gr}_i^{\Sigma}(G) \to \mathrm{gr}_i^{\Sigma'}(G')$ for each $0 \leq i \leq n-1$.

Problem 5. Let H be a group admitting a Jordan-Hölder series. Let $\ell(H)$ be the number of terms in a Jordan-Hölder series of H. Show this number is well defined.

Problem 6. Let G be a group and N be a normal subgroup of G. Prove that G has a Jordan-Hölder series if, and only if both N and G/N do. In that case, prove that $\ell(G) = \ell(N) + \ell(G/N)$.

Problem 7. Compute the derived and lower central series of the symmetric groups S_2, S_3 and S_4.

Problem 8. Assume that G is a (non-trivial) nilpotent group. Prove that $\mathsf{Z}(G) \neq \{e\}$. Here, $\mathsf{Z}(G)$ is the center of G.
(*Recall:* G is nilpotent if and only if G admits a composition series $G = H_0 \triangleright \ldots \triangleright H_m = \{e\}$ such that $[G, H_\ell] \subset H_{\ell+1}$ for every ℓ.)

Problem 9. Let n, k be positive integers and $p > 0$ a prime number. Compute Jordan-Hölder series for $\mathbb{Z}/p^k\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$.

Problem 10. Fix a finite simple group S. For a finite group G, choose a Jordan-Hölder series $\Sigma : G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = \{e\}$. Let $\mathrm{Mult}(S; G)$ be defined as:

$$\mathrm{Mult}(S; G) := \#\{j : G_j/G_{j+1} \cong S\}.$$

Prove that $\mathrm{Mult}(S; G)$ does not depend on the choice of the Jordan-Hölder series Σ.

Problem 11. Fix a finite simple group S. Let G be a finite group and $N \triangleleft G$ be a normal subgroup. Prove that $\mathrm{Mult}(S; G) = \mathrm{Mult}(S; N) + \mathrm{Mult}(S; G/N)$.