Problem 1. Prove or disprove: the ring $R := \mathbb{Z}[x]/(x^2 + 1)$ is a principal ideal domain, that is, $R$ is a domain, and every ideal of $R$ is generated by a single element.

Problem 2. Show that the ideal of $\mathbb{Z}[x]$ generated by 2 and $x$ is not principal.

Problem 3. (Radical of an ideal)
Fix a commutative ring $A$, and let $a \subseteq A$ be an ideal. Define
$$r(a) := \{ a \in A : a^n \in a \text{ for some } n \in \mathbb{Z}_{\geq 1} \}.$$ Prove the following statements for two ideals $a, b$ of $A$:

(i) $r(a)$ is an ideal of $A$.

(ii) $r(r(a)) = r(a)$.

(iii) $r(ab) = r(a \cap b) = r(a) \cap r(b)$.

(iv) $r(a + b) = r(r(a) + r(b))$.

Problem 4. (Ideal quotients) Let $a$ and $b$ be two ideals of a commutative ring $A$. Set:
$$\langle a : b \rangle := \{ a \in A : ab \subseteq a \}.$$ Prove the following statements:

(i) Prove that $\langle a : b \rangle$ is an ideal of $A$.

(ii) Compute the ideal quotient $\langle (n) : (m) \rangle$ in $\mathbb{Z}$.

Problem 5. Recall that the set of group endomorphisms of a group is a monoid under composition, with its unit the identity map.

(i) If $M$ is a left $R$-module, show that $\text{End}_{\text{Gps}}(M)$ is a ring, where addition and multiplication of maps $f, g \in \text{End}_{\text{Gps}}(M)$ is defined as follows:
$$(f + g)(m) = f(m) + g(m) \quad \text{for all } m \in M \quad \text{and} \quad f \cdot g := f \circ g.$$ (Hint: $\lambda(r)(m) = r \cdot m$ for each $m \in M$ and each $r \in R$.)

(ii) Show that this gives a shorter way to define left $R$-modules: A left $R$-module is an abelian group $M$ together with ring homomorphism $\lambda : R \to \text{End}_{\text{Gps}}(M)$. (Hint: $\lambda(r)(m) = r \cdot m$ for each $m \in M$ and each $r \in R$.)

(iii) Given a ring $R$, we define its opposite ring $R^{\text{op}}$ with multiplication $a \cdot b$ in $R^{\text{op}}$ equal to $b \cdot a$ in $R$. Show that a right $R$-module consists of an abelian group $N$ together with a ring homomorphism $\rho : R^{\text{op}} \to \text{End}_{\text{Gps}}(N)$ (Hint: $\rho(r)(n) = n \cdot r$ for each $m \in N$ and each $r \in R$.)
Problem 6. State and prove the three Isomorphism Theorems for modules over rings.

Problem 7. Let $R$ be a ring, and $M$ be a left $R$-module. Assume that we have $p \in \text{End}_R(M) := \text{Hom}_R(M, M)$ such that $p^2 := p \circ p = p$ (i.e., an idempotent map). Prove that $M \simeq M_1 \oplus M_2$, where $M_1$ is the kernel of $p$ and $M_2$ is the image of $p$.

Problem 8. Give a counterexample to the assertion of Problem 7, if we do not impose the condition $p^2 = p$.

Problem 9. Consider a collection $\{M_i\}_{i \in I}$ of submodules of a left $R$-module $M$, with the inclusion maps $f_i : M_i \hookrightarrow M$.

(i) Using the universal property for direct sums, show that we can build a natural map $f : \bigoplus_{i \in I} M_i \to M$ compatible with the inclusions $f_i$.

(ii) Show that $f$ is an isomorphism if and only if the following two conditions hold:

(a) $\sum_{i \in I} M_i$ (the submodule of $M$ generated by the set $\{M_i\}_{i \in I}$) equals $M$, and

(b) $M_i \cap \sum_{j \in I, j \neq i} M_j = \{0\}$ for all $i \in I$.

Problem 10. Given three left $R$-modules $M_1$, $M_2$ and $M_3$, we say that the short sequence

\begin{align*}
0 &\longrightarrow M_1 &\xrightarrow{f} & M_2 &\xrightarrow{g} & M_3 &\longrightarrow 0
\end{align*}

where $f$ and $g$ are $R$-linear maps is exact if $f$ is injective, $g$ is surjective and $\text{Im}(f) = \text{ker}(g)$. This sequence is trivial if we have an $R$-linear isomorphism $\eta : M_1 \oplus M_3 \to M_2$ satisfying $f = \eta \circ i : M_1 \to M_2$ and $\pi_2 = g \circ \eta : M_1 \oplus M_3 \to M_3$, where $i : M_1 \to M_1 \oplus M_3$ and $\pi_2 : M_1 \oplus M_3 \to M_3$ are the natural inclusion and projection maps. If these conditions holds, we say that both squares in the following diagram

\begin{align*}
0 &\longrightarrow M_1 &\xrightarrow{id} & M_1 &\xrightarrow{f} & M_2 &\xrightarrow{\eta} & M_3 &\xrightarrow{id} & 0 \\
0 &\longrightarrow M_1 &\xrightarrow{i} & M_1 \oplus M_3 &\xrightarrow{\pi_2} & M_3 &\longrightarrow 0
\end{align*}

commute.

(i) Show that the s.e.s. (1) is trivial if it admits a retraction $r : M_2 \to M_1$ (i.e. a group homomorphism with $r \circ f = id_{M_1}$) that is $R$-linear.

(ii) Show that the s.e.s. (1) is trivial if and only it splits, that is if there exists a section $s : M_3 \to M_2$ (i.e. a group homomorphism with $g \circ s = id_{M_3}$) that is $R$-linear. (Hint: Recall that for abelian groups, a s.e.s. splits if an only if it is trivial.)
In the next two problems, we take $A = \mathbb{Z}$, $M = \mathbb{Z}/m\mathbb{Z}$ and $N = \mathbb{Z}/n\mathbb{Z}$.

**Problem 11.** Given $\alpha \in M$, define $P_\alpha$ to be the abelian group generated by two elements $\{e_1, e_2\}$, subject to the following relations:

$$me_1 = 0 \quad \text{and} \quad ne_2 = \alpha e_1.$$ 

Verify that we have a natural short exact sequence:

$$\varepsilon_\alpha : 0 \rightarrow M \rightarrow P_\alpha \rightarrow N \rightarrow 0.$$

**Problem 12.** Determine the necessary and sufficient conditions for two short exact sequences $\varepsilon_\alpha$ and $\varepsilon_\beta$ from Problem 11 to be isomorphic (i.e. where we can build three vertical isomorphisms between the two sequences making the two squares commute, as seen in Problem 10).