ALGEBRA I (MATH 6111 AUTUMN 2021) - HOMEWORK 6

Problem 1. Prove or disprove: the ring $R := \mathbb{Z}[x]/(x^2+1)$ is a principal ideal domain, that is, R is a domain, and every ideal of R is generated by a single element.

Problem 2. Show that the ideal of $\mathbb{Z}[x]$ generated by 2 and x is not principal.

Problem 3. (Radical of an ideal)
Fix a commutative ring A, and let $\mathfrak{a} \subset A$ be an ideal. Define

$$r(\mathfrak{a}) := \{a \in A : a^n \in \mathfrak{a} \text{ for some } n \in \mathbb{Z}_{\geq 1}\}.$$

Prove the following statements for two ideals $\mathfrak{a}, \mathfrak{b}$ of A:

(i) $r(\mathfrak{a})$ is an ideal of A.

(ii) $r(r(\mathfrak{a})) = r(\mathfrak{a})$.

(iii) $r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$.

(iv) $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$.

Problem 4. (Ideal quotients) Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a commutative ring A. Set:

$$(\mathfrak{a} : \mathfrak{b}) := \{a \in A : a\mathfrak{b} \subseteq \mathfrak{a}\}.$$

(i) Prove that $(\mathfrak{a} : \mathfrak{b})$ is an ideal of A.

(ii) Compute the ideal quotient $((n) : (m))$ in $\mathbb{Z}$.

Problem 5. Recall that the set of group endomorphisms of a group is a monoid under composition, with its unit the identity map.

(i) If M is a left R-module, show that $\mathrm{End}_{\mathrm{Gps}}(M)$ is a ring, where addition and multiplicaion of maps $f, g \in \mathrm{End}_{\mathrm{Gps}}(M)$ is defined as follows:

$$(f+g)(m) = f(m) + g(m) \text{ for all } m \in M \quad \text{and} \quad f \cdot g := f \circ g.$$

(ii) Show that this gives a shorter way to define left R-modules: A left R-module is an abelian group M together with ring homomorphism $\lambda \colon R \to \mathrm{End}_{\mathrm{Gps}}(M)$. (*Hint:* $\lambda(r)(m) = r \cdot m$ for each $m \in M$ and each r in R.)

(iii) Given a ring R, we define its *opposite ring* R^{op} with multiplication $a \cdot b$ in R^{op} equal to $b \cdot a$ in R. Show that a right R-module consists of an abelian group N together with a ring homomorphism $\rho \colon R^{\mathrm{op}} \to \mathrm{End}_{\mathrm{Gps}}(N)$ (*Hint:* $\rho(r)(n) = n \cdot r$ for each $m \in N$ and each r in R.)

Problem 6. State and prove the three Isomorphism Theorems for modules over rings.

Problem 7. Let R be a ring, and M be a left R-module. Assume that we have $p \in \mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ such that $p^2 := p \circ p = p$ (i.e., an *idempotent* map). Prove that $M \simeq M_1 \oplus M_2$, where M_1 is the kernel of p and M_2 is the image of p.

Problem 8. Give a counterexample to the assertion of Problem 7, if we do not impose the condition $p^2 = p$.

Problem 9. Consider a collection $\{M_i\}_{i \in I}$ of submodules of a left R-module M, with the inclusion maps $f_i \colon M_i \hookrightarrow M$.

(i) Using the universal property for direct sums, show that we can build a natural map $f \colon \bigoplus_{i \in I} M_i \to M$ compatible with the inclusions f_i.

(ii) Show that f is an isomorphism if and only if the following two conditions hold:

(a) $\sum_{i \in I} M_i$ (the submodule of M generated by the set $\{M_i\}_{i \in I}$) equals M, and

(b) $M_i \cap \sum_{\substack{j \in I \\ j \neq i}} M_j = \{0\}$ for all $i \in I$.

Problem 10. Given three left R-modules M_1, M_2 and M_3, we say that the short sequence

$$\mathbf{0} \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow \mathbf{0} \tag{1}$$

where f and g are R-linear maps is *exact* if f is injective, g is surjective and $\mathrm{Im}(f) = \ker(g)$. This sequence is *trivial* if we have an R-linear isomorphism $\eta \colon M_1 \oplus M_3 \to M_2$ satisfying $f = \eta \circ i \colon M_1 \to M_2$ and $\pi_2 = g \circ \eta \colon M_1 \oplus M_3 \to M_3$, where $i : M_1 \to M_1 \oplus M_3$ and $\pi_2 : M_1 \oplus M_3 \to M_3$ are the natural inclusion and projection maps. If these conditions holds, we say that both squares in the following diagram

$$\begin{array}{ccccccccc} \mathbf{0} & \longrightarrow & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{g} & M_3 & \longrightarrow & \mathbf{0} \\ & & {\scriptstyle id}\uparrow & & {\scriptstyle \eta}\uparrow & & {\scriptstyle id}\uparrow & & \\ \mathbf{0} & \longrightarrow & M_1 & \xrightarrow{i} & M_1 \oplus M_3 & \xrightarrow{\pi_2} & M_3 & \longrightarrow & \mathbf{0} \end{array}$$

commute.

(i) Show that the s.e.s. (1) is trivial if it admits a retraction $r \colon M_2 \to M_1$ (i.e. a group homomorphism with $r \circ f = id_{M_1}$) that is R-linear.

(ii) Show that the s.e.s. (1) is trivial if and only it splits, that is if there exists a section $s \colon M_3 \to M_2$ (i.e. a group homomorphism with $g \circ s = id_{M_3}$) that is R-linear. (*Hint:* Recall that for abelian groups, a s.e.s. splits if an only if it is trivial.)

In the next two problems, we take $A = \mathbb{Z}$, $M = \mathbb{Z}/m\mathbb{Z}$ and $N = \mathbb{Z}/n\mathbb{Z}$.

Problem 11. Given $\alpha \in M$, define P_α to be the abelian group generated by two elements $\{e_1, e_2\}$, subject to the following relations:

$$m\, e_1 = 0 \qquad \text{and} \qquad n\, e_2 = \alpha\, e_1.$$

Verify that we have a natural short exact sequence:

$$\varepsilon_\alpha : \quad 0 \longrightarrow M \longrightarrow P_\alpha \longrightarrow N \longrightarrow 0\ .$$

Problem 12. Determine the necessary and sufficient conditions for two short exact sequences ε_α and ε_β from Problem 11 to be isomorphic (i.e. where we can build three vertical isomorphisms between the two sequences making the two squares commute, as seen in Problem 10).