ALGEBRA I (MATH 6111 AUTUMN 2021) - HOMEWORK 7

Problem 1. (Euclidean domains and Principal Ideal Domains)
Given an integral domain R, a *Euclidean function* on R is a map $f\colon R \smallsetminus \{0\} \to \mathbb{Z}_{\geq 0}$ satisfying the following property: "If $a, b \in R$ with $b \neq 0$, then there exist $q, r \in R$ such that $a = qb + r$ and either $r = 0$ or $f(r) < f(b)$."

A *Euclidean domain* is an integral domain that has at least one Euclidean function.

(i) Show that $\mathbb{Z}$ has Euclidean function $f(x) = |x|$.

(ii) Show that $R = \mathbb{K}[x]$ where $\mathbb{K}$ is a field with Euclidean function the degree of a polynomial.

(iii) Show that a Euclidean domain R is a principal ideal domain (that is, any ideal I of R is of the form $I = (x)$ for some $x \in R$).

Problem 2. Let R be a commutative ring, $x \in R$ be a nilpotent element (i.e. there exists $n \geq 1$ with $x^n = 0$), and $u \in R$ a unit (that is, u has a multiplicative inverse). Prove that $x + u$ is again a unit.

Problem 3. (Nilradical of a commutative ring) Consider a commutative ring R and let $\mathcal{N} \subset R$ be the set of all nilpotent elements. Show that $\mathcal{N}$ is an ideal of R.

Problem 4. Fix a commutative ring R and let $\mathfrak{a} \subset R$ be the set of all zero-divisors of R. Is $\mathfrak{a}$ an ideal of R?

Problem 5. Let A, B be commutative rings and let $f\colon A \to B$ be a ring homomorphism. Let $\mathfrak{p} \subset A$ be a prime ideal and define $\mathfrak{q} \subset B$ to be the ideal generated by $f(\mathfrak{p})$ in B. Prove or disprove: $\mathfrak{q}$ is a prime ideal.

Problem 6. Let R be a commutative ring and fix a prime ideal $\mathfrak{p}$ of R. Prove that $r(\mathfrak{p}^n) = \mathfrak{p}$ for every $n \geq 1$. (Here, $r(\cdot)$ denotes the radical ideal, as defined in Problem 3 of Homework 6.)

Problem 7. Let R be a commutative ring. Fix $\mathfrak{m} \subset R$ a maximal ideal, and let $\mathfrak{p}$ be a prime ideal. Assume that there exists $n \geq 1$ such that $\mathfrak{m}^n \subseteq \mathfrak{p}$. Prove that $\mathfrak{m} = \mathfrak{p}$.

Problem 8. (Jacobson radical)
Fix a commutative ring R and consider the following subset of R:

$$\mathfrak{J} := \{x \in R : 1 - xy \text{ is a unit for every } y \in R\}.$$

(i) Prove that $\mathfrak{J}$ is an ideal of R.

(ii) Prove that $\mathfrak{J} = \bigcap_{\substack{\mathfrak{m} \subset R \\ \mathfrak{m} \text{ maximal ideal}}} \mathfrak{m}$.

Problem 9. Consider the ring $R = \mathbb{C}[x]/(f)$, where $f \in \mathbb{C}[x]$ is a nonzero polynomial. Let us write $f(x) = \prod_{i=1}^{\ell}(x - a_i)^{n_i}$, where $a_1, \dots, a_\ell \in \mathbb{C}$ and $n_1, \dots, n_\ell \in \mathbb{Z}_{\geq 1}$. Prove that

$$R \xrightarrow{\sim} \prod_{i=1}^{\ell} \left(\mathbb{C}[x]/((x - a_i)^{n_i})\right) .$$

Problem 10. (Generalized prime avoidance)
Let R be a commutative ring and let $S \subset R$ be a set that is closed under multiplication and addition. Let $\mathfrak{p}_i \subset R$ for $i = 1, \dots, n$, be a a finite list of ideals of R where at most two of them are not prime. Prove that if $S \subset \bigcup_{i=1}^n \mathfrak{p}_i$, then there exists $j = 1, \dots, n$ with $S \subset \mathfrak{p}_j$. (*Hint:* Review the proof of the prime avoidance theorem.)

Problem 11. Fix a prime $p \geq 2$ and consider the set

$$R = \{\frac{a}{b} \in \mathbb{Q} : p \nmid b \text{ with } \gcd(a, b) = 1\}$$

(i) Show that R is a subring of $\mathbb{Q}$.

(ii) Show that R has a unique maximal ideal, so it is a local ring.

Problem 12. (Ring of fractions relative to S)
Let R be a commutative ring, $S \subseteq R$ a multiplicatively closed set. We write a representative for the equivalence class (r, s) in $S^{-1}R := R \times S/\sim$ by $\frac{r}{s}$.

(i) Show that the addition and multiplication operations on $S^{-1}R$ are well-defined and give $S^{-1}R$ a ring struture.

(ii) Show that the map $j_S : R \to S^{-1}R$ with $j_S(r) = \frac{r}{1}$ is a ring homomorphism, and $j_S(t)$ is invertible in $S^{-1}R$ for each $t \in S$.

(iii) Show that $S^{-1}R$ satisfies the following universal property. Given a commutative ring B and a ring homomorphism $f : R \to B$ where $f(s)$ is invertible in B for each $s \in S$, then there exists a unique ring homomorphism $\tilde{f} : S^{-1}R \to B$ with $f = \tilde{f} \circ j_S$. Moreover, $\tilde{f}(\frac{r}{s}) = f(r)\, f(s)^{-1}$. (*Hint:* You must show that this map is well-defined, i.e. it is independent of the choice of representative for $\frac{r}{s}$)

Problem 13. If $R = \mathbb{Z}$ and $S = R \smallsetminus \{0\}$ show that $S^{-1}R \simeq \mathbb{Q}$.

Problem 14. Consider $R = \mathbb{Z}/6\mathbb{Z}$, and let $S = \{1, 2, 4\}$. Show that S is multiplicatively closed and compute $S^{-1}R$. How many elements does $S^{-1}R$ have?

Problem 15. Let $\mathbb{K}$ be a filed and let $R = \mathbb{K}[x]$. Show that the set $S = \{1, x^n : n \geq 1\}$ is multiplicatively closed, and $S^{-1}R \simeq \mathbb{K}[x, x^{-1}]$ (the Laurent polynomial ring over $\mathbb{K}$).

Problem 16. (**Modules of fractions**)

Let R be a commutative ring, $S \subseteq R$ a multiplicatively closed set and M an R-module. Consider the relation $\sim$ defined on $M \times S$:

$$(m, s) \sim (m', s') \quad \text{if, and only if,} \quad \exists\, t \in S \text{ with } t(s'\, m - s\, m') = 0 \in M.$$

(i) Show that $\sim$ defines an equivalence relation on $M \times S$. We write the equivalence class of (m, s) as $\frac{m}{s}$.

(ii) Show that $S^{-1}M := (M \times S)/\sim$ is an $S^{-1}R$-module with operations:

$$\frac{m}{s} + \frac{m'}{s'} = \frac{s'\, m + s\, m'}{ss'} \quad \text{and} \quad \frac{a}{t}\,\frac{m}{s} = \frac{a\, m}{ts}.$$

(iii) Show that the map $i_S \colon M \to S^{-1}M$ with $i_S(m) = \frac{m}{1}$ is R-linear.

(iv) For each $s \in S$, consider the map $\rho(s) \colon S^{-1}M \to S^{-1}M$ sending $\frac{m}{t} \mapsto s\frac{m}{t}$ for each m in $S^{-1}M$. Show that $\rho(s)$ is R-linear and $\rho(s)$ is an isomorphism of R-modules, with $\rho(s)^{-1}(\frac{m}{t}) = \frac{m}{st}$ for all $m \in S^{-1}M$.

(v) Prove that $S^{-1}M$ satisfies the following universal property. Given an R-module N where the multiplication maps $\rho(s) \colon N \to N$ are invertible (as an R-linear map) for each $s \in S$, and an R-linear map $f \colon M \to N$, then, there exists a unique R-linear map $\tilde{f} \colon S^{-1}M \to N$ satisfying $\tilde{f} \circ i_S = f$. Moreover, $\tilde{f}(\frac{m}{s}) = \rho(s)^{-1}(f(m))$.

Problem 17. Let R be a commutative ring, $S \subset R$ a multiplicatively closed set and $\mathfrak{a} \subseteq R$ be an ideal. Prove that $S^{-1}\mathfrak{a}$ (the module of fractions of $\mathfrak{a}$ relative to S defined in Problem 16) is the ideal in $S^{-1}R$ generated by $j_S(\mathfrak{a})$, where $j_S \colon R \to S^{-1}R$ is the natural ring homomorphism.