ALGEBRA I (MATH 6111 AUTUMN 2021) - HOMEWORK 8

In all problems below, we assume R is a commutative ring.

Problem 1. Let R be a commutative ring, and let $S \subseteq R$ be a multiplicatively closed set. Let $\mathfrak{p}$ be an ideal which is maximal among the ideals in R not intersecting S. That is, maximal with respect to inclusion, from the following set:

$$I_S = \{\mathfrak{a} \subseteq R : \mathfrak{a} \text{ is an ideal of } R, \mathfrak{a} \cap S = \emptyset\}.$$

Prove that $\mathfrak{p}$ is prime.

Problem 2. (Nilradical of a commutative ring) Consider a commutative ring R and let $\mathcal{N} \subset R$ be the set of all nilpotent elements (see Problem 3 of Homework 7). Prove that

$$\mathcal{N} = \bigcap_{\substack{\mathfrak{p} \subset R \\ \mathfrak{p} \text{ prime ideal}}} \mathfrak{p}.$$

(*Hint:* Use Problem 1 for the inclusion $(\supseteq)$.)

Problem 3. Consider the ring $R[x,y]$ and the multiplicatively closed set S generated by x, that is $S = \{1, x, x^2, x^3, \ldots\}$. Prove the following isomorphisms of rings

$$(S^{-1})(R[x,y]/(xy)) \simeq R[x, x^{-1}].$$

Problem 4. Consider the following sequence of R-linear maps between R-modules:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

(i) Prove that this sequence is exact if, and only if, for every maximal ideal $\mathfrak{m} \subset R$, the following sequence of $A_{\mathfrak{m}}$-modules is exact:

$$0 \longrightarrow M'_{\mathfrak{m}} \longrightarrow M_{\mathfrak{m}} \longrightarrow M''_{\mathfrak{m}} \longrightarrow 0.$$

(ii) Show that the same is true if we consider localizations at all prime ideals $\mathfrak{p}$ of R.

Problem 5. Given an R-module M, we define the *support* of M as:

$$\text{Supp}(M) = \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal of } R, M_{\mathfrak{p}} \neq (0)\}.$$

(i) If N is a submodule of M, show that $\text{Supp}(M) = \text{Supp}(N) \cup \text{Supp}(M/N)$.

(ii) Given an ideal $\mathfrak{a}$ of R, show that

$$\text{Supp}(R/\mathfrak{a}) = \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal of } R \text{ with } \mathfrak{p} \supseteq \mathfrak{a}\}.$$

Problem 6. Assume R is a domain and let $\mathfrak{m}$ and $\mathfrak{n}$ be maximal ideals of R. Consider the prime ideals $R_{\mathfrak{n}}\mathfrak{m} \subset R_{\mathfrak{n}}$ and $R_{\mathfrak{m}}\mathfrak{n} \subset R_{\mathfrak{m}}$ in the corresponding localizations. Prove or disprove: $(R_{\mathfrak{m}})_{R_{\mathfrak{m}}\mathfrak{n}}$ and $(R_{\mathfrak{n}})_{R_{\mathfrak{n}}\mathfrak{m}}$ are isomorphic (local) rings. What happens if R is not a domain?

Problem 7. Let $\mathfrak{p} \subsetneq R$ be a prime ideal of R. Show that the quotient $R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})$ is isomorphic to the quotient field $\operatorname{Quot}(R/\mathfrak{p})$.

Problem 8. The goal of this exercise is to use exactness of localization to reprove that

$$R = \bigcap_{\substack{\mathfrak{p}\subseteq R \\ \mathfrak{p} \text{ prime}}} R_{\mathfrak{p}} = \bigcap_{\substack{\mathfrak{m}\subseteq R \\ \mathfrak{m} \text{ maximal}}} R_{\mathfrak{m}}$$

if R is a domain. We can show that $R = \bigcap_{\substack{\mathfrak{m}\subseteq R \\ \mathfrak{m} \text{ maximal}}} R_{\mathfrak{m}}$ by working with localizations of modules.

(i) View $M := \bigcap_{\substack{\mathfrak{m}\subseteq R \\ \mathfrak{m} \text{ maximal}}} R_{\mathfrak{m}}$ as an R-module and show that R is a submodule of M. In particular, we can consider the quotient module $M' := M/R$.

(ii) Given any maximal ideal $\mathfrak{n}$ of R, use the exactness of localization to certify that $(M')_{\mathfrak{n}} \simeq M_{\mathfrak{n}}/R_{\mathfrak{n}}$ as $R_{\mathfrak{n}}$-modules.

(iii) Next, show that $M_{\mathfrak{n}} = \bigcap_{\substack{\mathfrak{m}\subseteq R \\ \mathfrak{m} \text{ maximal}}} (R_{\mathfrak{m}})_{R_{\mathfrak{m}}\mathfrak{n}}$.

(iv) Use that $R_{\mathfrak{n}}$ is a local ring with maximal ideal $R_{\mathfrak{n}}\mathfrak{n}$ to show that

$$\bigcap_{\substack{\mathfrak{m}\subseteq R \\ \mathfrak{m} \text{ maximal}}} (R_{\mathfrak{m}})_{R_{\mathfrak{m}}\mathfrak{n}} \subseteq (R_{\mathfrak{n}})_{R_{\mathfrak{n}}\mathfrak{n}} = R_{\mathfrak{n}}.$$

(v) The previous three items ensure that all localizations $(M')_{\mathfrak{n}}$ are trivial. Conclude from here that $M' = 0$.

Problem 9. Let B be a commutative ring which contains R as a subring. Assume that B is finitely generated as a ring over R, that is, there exists finitely many elements $b_1, \dots, b_\ell$ in B such that every $b \in B$ can be written as a polynomial expression $\{b_1, \dots, b_\ell\}$ with R coefficients. Prove that if R is Noetherian, then so is B.

Problem 10. Assume $R[x]$ is Noetherian. Does it imply that R is Noetherian?

Problem 11. Let M be a Noetherian module over R. Consider the set $M[x]$, defined as:

$$M[x] = \{\sum_{i=0}^{N} m_i x^i : m_i \in M \text{ for all } i, N \geq 0\}.$$

(i) Show that $M[x]$ is an $R[x]$-module.

(ii) Show that $M[x]$ is a Noetherian module over $R[x]$.

(iii) What happens when we view $M[x]$ as an R-module?

Problem 12. Assume that R is not Noetherian. Let $\mathcal{S}$ be the set of all ideals of R which are not finitely generated. Prove that this set has a maximal element and that any such maximal element is necessarily a prime ideal of R.

Problem 13. Assume that R is Noetherian. Show that the nilradical $\mathcal{N}$ is a nilpotent ideal (i.e., there exists $k \in \mathbb{Z}_{\geq 1}$ with $\mathcal{N}^k = (0)$.)

Problem 14. Assume that $R_\mathfrak{p}$ is a Noetherian ring, for every prime ideal $\mathfrak{p}$ of R. Prove or disprove: R is Noetherian.

Problem 15. Let M be a Noetherian module over R. Let $f\colon M \to M$ be a surjective R-linear map. Prove that f is an isomorphism. (*Hint:* consider the chain of submodules $\{\mathrm{Ker}(f^n)\}_{n\geq 0}$, where $f^0 = \mathrm{id}_M$.)