ALGEBRA I (MATH 6111 AUTUMN 2021) - HOMEWORK 9

In all problems below, we assume R is a commutative ring.

Problem 1. Let $I \subsetneq R$ a proper ideal, and $\mathfrak{p} \subsetneq R$ a prime ideal such that $I \subset \mathfrak{p}$. Prove that $r(I) \subset \mathfrak{p}$.

Problem 2. Show that if $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_n\}$ are pairwise coprime ideals of R (that is, $\mathfrak{a}_i + \mathfrak{a}_j = R$ for each $i \neq j$) then:

$$\bigcap_{i=1}^{n} \mathfrak{a}_i = \mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_n.$$

(*Hint:* Show that $\mathfrak{a}_n$ and the product ideal $\mathfrak{a}_1 \ldots \mathfrak{a}_{n-1}$ are coprime and induct on $n \geq 2$. The case $n = 2$ was proven in Lecture 26)

Problem 3. Assume that R is Noetherian. Prove that so is $R[[x]]$ (i.e., power series with coefficients over R).

Problem 4. Assume that R is Noetherian. For any ideal I, prove that there exists $n \geq 1$ such that $(r(I))^n \subset I$.

Problem 5. Assume $(R, \mathfrak{m})$ is a local Noetherian ring of has dimension zero (so each prime ideal of R is maximal) with $\mathfrak{m}^n = (0)$ for some $n \geq 1$. This problem shows that R is Artinian (completing the proof given in class).

(i) Consider the successive quotients $M_j := \mathfrak{m}^j/\mathfrak{m}^{j+1}$ for each $j = 0, \ldots, n$. They are vector spaces over the field $k = R/\mathfrak{m}$. Show that they have finite dimension (*Hint:* They are also R-modules).

(ii) Fix any descending chain of ideals of R, and consider the $n+1$ chains of ideals obtained by intersecting each member of the chain with $\mathfrak{m}^j$ for $j = 0, \ldots, n$. Show that these chains stabilize (*Hint:* For each $j < n$, consider the successive quotients and produce a descending chain of k-vector spaces that stabilizes. Show that these vector spaces have finite dimension. Notice that for $j = n$, the chain consists of only the zero ideal.)

(iii) Use descending induction on $j = 0, \ldots, n$ and the previous item to conclude the original descending chain of ideals stabilizes.

(iv) In particular for $j = 0$, the previous item confirms the descending chain condition holds for R, so R is Artinian.

Problem 6. Prove that a finite direct product of Artinian rings is Artinian. (*Hint:* Show that the ideals of such a product are products of ideals.)

Problem 7. Let R be an Artinian ring. Show that the nilradical $\mathcal{N}$ is a nilponent ideal, that is, we can find an integer $n \geq 0$ with $\mathcal{N}^n = (0)$. (*Hint:* Consider the descending chain of ideals $\mathcal{N} \supseteq \mathcal{N}^2 \supseteq \mathcal{N}^3 \supseteq \ldots$ and let $\mathfrak{a}$ be the ideal where the chain stabilizes. Assuming $\mathfrak{a} \neq (0)$, consider the set $\mathscr{S}$ of ideals $\mathfrak{b}$ of R with $\mathfrak{a}\mathfrak{b} \neq (0)$. Use the minimal element I of $\mathscr{S}$ to find elements $x \in I$ and $y \in \mathfrak{a}$ with $x = xy$. This will lead you to a contradiction.)

Problem 8. Let M be an Artinian R-module. That is, for every descending chain of submodules $M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$, there exists $\ell \geq 0$ such that $M_\ell = M_{\ell+1} = \ldots$. Assume there is an injective R-linear map $f\colon M \to M$. Prove that f is an isomorphism.

Problem 9. Show that an ideal $\mathfrak{q} \subset R$ is primary if, and only if, every zero divisor in $R/\mathfrak{q}$ is nilpotent.

Problem 10. Let $R = \mathbb{K}[x, y]$ where $\mathbb{K}$ is any field and let $\mathfrak{q} = (x, y^2)$.

(i) Show that $r(\mathfrak{q}) = (x, y)$.

(ii) Show that $\mathfrak{q}$ is a primary ideal using the definition (i.e., for any product $fg \in \mathfrak{q}$ with $g \notin \mathfrak{q}$ we can find $n \geq 1$ with $f^n \in \mathfrak{q}$).

(iii) Show that $(x, y)^2 \subsetneq \mathfrak{q}$.

Problem 11. Consider the quotient ring $R = \mathbb{K}[x, y, y]/(xy - z^2)$ and the ideal $\mathfrak{p}$ of R with generators $\{\bar{x}, \bar{y}\}$.

(i) Show that $\mathfrak{p}$ is a prime ideal (*Hint:* Show that the quotient $R/\mathfrak{p}$ is a domain).

(ii) Show that $\mathfrak{p}^2$ is not primary by showing that $\bar{z}^2 = \bar{y}\bar{x} \in \mathfrak{p}^2$ and $\bar{x} \notin \mathfrak{p}^2$, but $y \notin r(\mathfrak{p})$.

(iii) Show that $\bar{y} \notin \mathfrak{p}^2$ and $x^2 \in \mathfrak{p}^2$. *Hence, the definition of primary ideal is not symmetric.*

Problem 12. Let $\mathfrak{q} \subset R$ be an ideal. If its radical ideal $r(\mathfrak{q})$ is maximal, then show that $\mathfrak{q}$ is primary.

Problem 13. Let $\mathfrak{m} \subset R$ be a maximal ideal. Prove that $\mathfrak{m}^n$ is primary for every $n \geq 1$.

Problem 14. Prove that $(4, t) \subset \mathbb{Z}[t]$ is a primary ideal. Verify that $r((4, t)) = (2, t)$ which is a maximal ideal in $\mathbb{Z}[t]$. Prove that $(2, t)^2 \subsetneq (4, t) \subsetneq (2, t)$. *Hence, a primary ideal need not be power of a prime.*

Problem 15. Let $R = \mathbb{K}[x, y]$, $I = (x^2, xy) \subset R$. Take $\mathfrak{p} = (x)$ and $\mathfrak{q}_n = (x^2, xy, y^n)$ for each $n \geq 2$. Prove that

(i) $\mathfrak{p}$ is a prime ideal. Each $\mathfrak{q}_n$ is primary and $r(\mathfrak{q}_n) = (x, y)$.
(ii) $\mathfrak{p} \cap \mathfrak{q}_n = I$.

Hence, we have infinitely many distinct primary decompositions. Notice that they all have the same set of primes $\{(x), (x, y)\}$.