ALGEBRA I (MATH 6111 AUTUMN 2021) - HOMEWORK 10

In all problems below, we assume R is a commutative ring and $\mathbb{K}$ is a field.
If R is a PID, and M is an R-module, the *rank* of M is defined as the rank of the free module M/M_{tor} where M_{tor} is the submodule of M consisting of all torsion elements of M.

Problem 1. Let $\mathfrak{a} \subset R$ be an ideal. Prove that

$$r(\mathfrak{a}) = \bigcap_{\mathfrak{p} \in \text{Min}(\mathfrak{a})} \mathfrak{p}.$$

Problem 2. Assume R is Noetherian and let $\mathfrak{p} \subset R$ be a prime ideal. Prove that $R_{\mathfrak{p}}$ is Artinian if, and only if $\mathfrak{p}$ is a minimal prime ideal of R.

Problem 3. Consider a primary ideal $\mathfrak{q}$ in R with radical $r(\mathfrak{q}) = \mathfrak{p}$. Let $S \subsetneq R$ be a multiplicatively closed set.

(i) Prove that $S \cap \mathfrak{q} \neq \emptyset$ if, and only if, $S \cap \mathfrak{p} \neq \emptyset$.

(ii) Assume that $S \cap \mathfrak{p} = \emptyset$. Show that $S^{-1}\mathfrak{q}$ is a primary ideal in $S^{-1}R$ and $r(S^{-1}\mathfrak{q}) = S^{-1}\mathfrak{p}$ in $S^{-1}R$.

Problem 4. Prove the following equality is true in the ring $\mathbb{K}[x, y]$, where $\mathbb{K}$ is any field.

$$(x^2, y) \cap (x, y^2) = (x, y)^2.$$

Prove that $(x, y)^2 \subset \mathbb{K}[x, y]$ is a primary ideal. (*Hence, primary does not imply irreducible.*)

Problem 5. Assume R is an integral domain. Prove that an ideal $\mathfrak{a}$ is free as an R-module if and only if $\mathfrak{a}$ is principal (i.e. admits one generator).

Problem 6.

Let R be an integral domain and let $\mathbb{K}$ be the field of fractions of R.

(i) Let M be a finitely generated R-module and let V be the vector space over $\mathbb{K}$ obtained from M by extension of scalars (i.e. $V = S^{-1}M$ where $S = R \setminus \{0\}$). Prove that the rank of M is equal to the dimension of V over $\mathbb{K}$.

(ii) If M is a free R-module, show that any two maximal linear independent subsets of M have the same cardinality.

Problem 7. Assume that for all finitely generated free modules M over R with rank n we have that every submodule of M is free of rank $\leq n$. Prove that R is a PID.

Problem 8. Prove that $(\mathbb{Q}_{>0}, *)$ is a free $\mathbb{Z}$-module and determine a basis for it.

Problem 9. Fix a module M over R and let $T\colon M \to M$ be an R-linear map. Prove that M is a module over $R[x]$ with scalar multiplication $f(x) \cdot m = f(T)(m)$ for all m in M.

Problem 10. Let $\mathbb{K}$ be a field and $g(x) \in \mathbb{K}[x] \smallsetminus \{0\}$. Show that $\mathbb{K}[x]/(g(x))$ is a $\mathbb{K}$-vector space of dimension $\deg(g)$.

Problem 11. Prove or disprove:

(i) $(\mathbb{Q}, +)$ is a free $\mathbb{Z}$-module; (*Hint:* There are non-zero $\mathbb{Z}$-linear maps from any free $\mathbb{Z}$-module to $\mathbb{Z}$.)

(ii) $\mathbb{K}(x)$ is a free $\mathbb{K}[x]$-module for any field $\mathbb{K}$.

Problem 12. Assume $M_1, \ldots, M_r$ are R-modules and let $N_i \subset M_i$ be submodules. Show that:

$$\frac{\bigoplus_{i=1}^r M_i}{\bigoplus_{i=1}^r N_i} \simeq \bigoplus_{i=1}^r \frac{M_i}{N_i}.$$

Problem 13. Consider a PID R and let $\mathbf{v} = (a_1, \ldots, a_n)$ be a vector in R^n. Prove that we can extend $\mathbf{v}$ to a basis of the free module R^n if and only if the ideal generated by $\{a_1, \ldots, a_n\}$ is the unit ideal.

Problem 14. (Modules over non commutative rings) The following exercise provides an example of a non-commutative ring A for which $A^n \simeq A^m$ for all $m, n \in \mathbb{N}$.

Let V be an infinite-dimensional vector space over $\mathbb{R}$ with a countable basis $\{v_n \colon n \in \mathbb{N}\}$. Let $A = \operatorname{End}_{\mathbb{R}} V$. Let $T, T' \in A$ be defined by $T(v_{2n}) = v_n$, $T(v_{2n-1}) = 0$, $T'(v_{2n}) = 0$, $T'(v_{2n-1}) = v_n$ for all $n \geq 1$. Prove that $\{T, T'\}$ is a basis for A as a left A-module. Thus, $A \simeq A^2$. Prove that $A^n \simeq A^m$ for any $m, n \in \mathbb{N}$.

Problem 15. (Free modules with infinite rank)
Let R be a PID and let M be a free R-module. Let $F \neq (0)$ be a submodule of M. We aim to show that F is free. Let $\{v_i\}_{i \in I}$ be a basis for M. For each $J \subseteq I$, let $M_J = R(v_j : j \in J)$.

(i) Consider the sets $F_J = F \cap M_J$ where $J \subseteq I$, and the set of triples

$$S = \{(F_J, J', w) : J \subseteq I, F_J \text{ is free }, w\colon J' \to F_J \text{ is a basis for } F_J \text{ indexed by } J' \subseteq J\}.$$

Show that S is a non-empty set.

(ii) Show that the following relation $\leq$ on S defines a partial order on S:

$$(F_J, J', w) \leq (F_K, K', u) \text{ if } J \subseteq K, J' \subseteq K' \text{ and } u_{|_{J'}} = w.$$

In other words, the basis u for F_K is an extension of the basis w for F_J.

(iii) Use Zorn's Lemma on S and show that a maximal element of S has $J = I$, so $F_J = F$. (*Hint:* Use the technique in the proof of Theorem 2 of Lecture 30 (the case where $\operatorname{rank}(M) < \infty$).)

(iv) Conclude that there exists a basis for F indexed by a subset of I (so the rank of a module is well-defined and $\operatorname{rank}(F) \leq \operatorname{rank}(M)$.)

Problem 16. If R is a PID, and M, N are finitely generated R-modules of rank m and n respectively, prove that $M \oplus N$ is a finitely generated R-module of rank $m + n$. Describe the torsion component of $M \oplus N$.

Problem 17. Let $R = \mathbb{K}[X, Y]$ be a polynomial ring in two variables over $\mathbb{K}$. Give an example of a module over R, which is finitely generated and torsion free, but not free. Do the same for $R = \mathbb{Z}[X]$.