In all problems below, we assume $R$ is a commutative ring and $\mathbb{K}$ is a field.
Two $n \times n$ matrices $A$ and $C$ over $\mathbb{K}$ are similar if $A = G^{-1}CG$ for some $G \in \text{GL}_n(\mathbb{K})$

**Problem 1.** Let $R$ be a PID, $a \in R \setminus \{0\}$ and $M = R/(a)$. Let $p$ be a prime of $R$ dividing $a$, and let $n$ be the highest power of $p$ dividing $a$. Prove that

$$p^{k-1}M/p^kM \simeq \begin{cases} R/(p) & \text{for } k = 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

**Problem 2.** Let $T$ be the linear operator on $V = \mathbb{C}^2$ whose matrix is \[
\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \]. Is the corresponding $\mathbb{C}[X]$-module cyclic?

**Problem 3.** Classify all abelian groups of order 32, 36 and 200.

**Problem 4.** Prove or disprove:
(i) $(\mathbb{Z}/8\mathbb{Z})^* \simeq (\mathbb{Z}/3\mathbb{Z})^* \times (\mathbb{Z}/3\mathbb{Z})^*$;

(ii) $(\mathbb{Z}/16\mathbb{Z})^* \simeq (\mathbb{Z}/3\mathbb{Z})^* \times (\mathbb{Z}/5\mathbb{Z})^*$;

**Problem 5.** (Diagonalization of matrices) Prove that $A \in \text{Mat}_{n \times n}(\mathbb{K})$ is diagonalizable over $\mathbb{K}$ (i.e., similar to a diagonal matrix), if and only the minimal polynomial of $A$ has no repeated roots.

**Problem 6.** Find all possible rational normal forms and Jordan forms of a matrix $A$ whose characteristic polynomial is $(X + 2)^2(X - 5)^3$.

**Problem 7.** Find all possible rational normal forms and Jordan forms of $8 \times 8$ matrices over $\mathbb{C}$ whose minimal polynomial is $X^2(X - 1)^3$.

**Problem 8.** If $N$ is a $k \times k$ nilpotent matrix such that $N^k = 0$ but $N^{k-1} \neq 0$, prove that $N$ is similar to its transpose. (Hint: Prove it for a Jordan block matrix $N$ of size $k$ by finding a permutation matrix $P$ with $N^T = PNP^{-1}$)

**Problem 9.** Prove that two $2 \times 2$ matrices over $\mathbb{K}$ are similar if and only if they have the same minimal polynomial.

**Problem 10.** Prove that two $3 \times 3$ matrices over $\mathbb{K}$ are similar if and only if they have the same minimal and characteristic polynomials.
Problem 11. Prove the Cayley-Hamilton Theorem over any commutative ring $R$: Let $A \in \text{Mat}_{n \times n}(R)$. If $f(X) = \det(XI_n - A)$, then $f(A) = 0$.

(Hint: Use the identity $\text{Cof}(B)B = B\text{Cof}(B) = \det(B)I_n$ for any $n \times n$ matrix $B$, where $\text{Cof}(B)$ is the cofactor matrix of $B$).

Problem 12. Let $A, B$ be two $n \times n$ matrices over a field $\mathbb{K}$. Assume that $AB = BA$ and both $A$ and $B$ are diagonalizable (i.e. has a basis of eigenvectors). Show that $A$ and $B$ can be simultaneously diagonalized. (Hint: By induction on $n$. For the inductive step you will need to show that any eigenspace $E_\lambda(A) = \{v \in \mathbb{K}^n : Av = \lambda v\}$ is invariant under $B$.)

Problem 13. (Nakayama’s Lemma v2)
Consider a finitely generated $R$-module $M$ and an ideal $I$ of $R$ included in the Jacobson radical of $R$ (see Problem 8 on Homework 7). If $IM = M$, prove that $M = 0$.

Problem 14. (Nakayama’s Lemma v3)
Consider a finitely generated $R$-module $M$ and an ideal $I$ of $R$. If $M = IM$, then prove that there exists an element $a \in I$ with $m = am$ for all $m \in M$ (equivalently, $(1-a)M = 0$).

(Hint for Problems 13 and 14: Follow the proof of Nakayama’s lemma discussed in Lecture 34.)

Problem 15. (Nakayama’s Lemma v3 fails over non-commutative rings)
Consider a non-commutative ring $A$ with no zero divisors and let $I$ be a f.g. proper non-zero idempotent ideal (i.e., $I^2 = I$). Show that Nakayama’s Lemma v3 from Problem 14 fails for $M = I$.

(Note: Examples of such rings arise in Lie Theory. We can take $g$ to be a perfect Lie algebra and let $I = gA$ be the augmentation ideal of its enveloping algebra $A = U(g)$)

Problem 16. (An application of Nakayama’s lemma due to Vasconcelos)
The goal of this exercise is to extend Problem 15 of Homework 8 to the non-Noetherian case. More precisely, assume $M$ is a finitely generated $R$-module and let $f : M \to M$ be a surjective $R$-linear map. We wish to show that $f$ is injective, and hence an isomorphism.

(i) Show that $M$ becomes an $R[x]$ mode via $P(x) \cdot m = P(f)(m)$.

(ii) Show that $M$ is a finitely generated $R[x]$-module with the structure defined in (i).

(iii) Show that ideal $I = (x) \subset R[x]$ satisfies $IM = M$.

(iv) Use Problem 14 to find a polynomial $P(x) \in I$ with $m = P(x)m$ for all $m \in M$.

(v) Use item (iv) to conclude that $\text{Ker}(f) = \{0\}$.

Problem 17. Show that bases of vector spaces of any dimension (defined as linearly independent spanning sets) are maximal linearly independent sets.
Problem 18. Consider a $\mathbb{K}$-vector space $V$ and let $W \subset V$ be a subspace. Show that:

$$(V/W)^* = \{ \xi \in V^* : \xi|_W = 0 \}.$$

Problem 19. Consider a collection $\{V_i : i \in I\}$ of finite-dimensional $\mathbb{K}$-vector spaces and let $W$ be a vector space.

(i) Prove that $\text{Hom}_\mathbb{K}(\bigoplus_{i \in I} V_i, W) \cong \prod_{i \in I} \text{Hom}_\mathbb{K}(V_i, W)$.

(ii) Prove that $(\bigoplus_{i \in I} V_i)^* \cong \prod_{i \in I} (V_i)^*$.

(In both items, direct products/sums of vector spaces are defined as the corresponding operations on $\mathbb{K}$-modules.)

Problem 20. Let $V$ and $W$ be two finite-dimensional $\mathbb{K}$ vector spaces, with bases $B_V$ and $B_W$, respectively. Assume $\dim V = n$ and $\dim W = m$. Consider any linear transformation $f : V \to W$ and its corresponding dual map $f^* : W^* \to V^*$.

(i) Show that if $A = [f]_{B_V,B_W}$ is the $m \times n$ matrix representative of $f$ with respect to the bases $B_V$ and $B_W$, then $A^T = [f^*]_{(B_W)^*,(B_V)^*}$ is the matrix representative of $f^*$ with respect to the dual bases $(B_W)^*$ and $(B_V)^*$.

(ii) Show that $W^* \cong (\text{Im } f)^* \oplus \text{Ker}(f^*)$, where we view $(\text{Im } f)^* \subset W^*$ by extending a basis of $\text{Im } f$ to a basis of $W$ and taking duals.

(iii) Show that the images of $f$ and $f^*$ have the same dimension, and conclude from this that the (column) ranks of $A$ and $A^T$ agree.

Problem 21. Given a $\mathbb{K}$-vector space $V$, consider the dual vector space to $V^*$. We denote it by $(V^*)^*$ and call it the “double dual” of $V$. Show that we can view $V$ as a subspace of its double dual by writing an explicit injective linear map $V \hookrightarrow (V^*)^*$. If $V$ is finite dimensional, conclude that $V \cong (V^*)^*$. 