ALGEBRA I (MATH 6111 AUTUMN 2021) - HOMEWORK 11

In all problems below, we assume R is a commutative ring and $\mathbb{K}$ is a field.
Two $n \times n$ matrices A and C over $\mathbb{K}$ are *similar* if $A = G^{-1}CG$ for some $G \in \mathrm{GL}_n(\mathbb{K})$

Problem 1. Let R be a PID, $a \in R \setminus \{0\}$ and $M = R/(a)$. Let p be a prime of R dividing a, and let n be the highest power of p dividing a. Prove that

$$p^{k-1}M/p^kM \simeq \begin{cases} R/(p) & \text{for } k = 1, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

Problem 2. Let T be the linear operator on $V = \mathbb{C}^2$ whose matrix is $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. Is the corresponding $\mathbb{C}[X]$-module cyclic?

Problem 3. Classify all abelian groups of order 32, 36 and 200.

Problem 4. Prove or disprove:

(i) $(\mathbb{Z}/8\mathbb{Z})^* \simeq (\mathbb{Z}/3\mathbb{Z})^* \times (\mathbb{Z}/3\mathbb{Z})^*$;

(ii) $(\mathbb{Z}/16\mathbb{Z})^* \simeq (\mathbb{Z}/3\mathbb{Z})^* \times (\mathbb{Z}/5\mathbb{Z})^*$;

Problem 5. (Diagonalization of matrices) Prove that $A \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ is diagonalizable over $\mathbb{K}$ (i.e., similar to a diagonal matrix), if and only the minimal polynomial of A has no repeated roots.

Problem 6. Find all possible rational normal forms and Jordan forms of a matrix A whose characteristic polynomial is $(X + 2)^2(X - 5)^3$.

Problem 7. Find all possible rational normal forms and Jordan forms of 8×8 matrices over $\mathbb{C}$ whose minimal polynomial is $X^2(X - 1)^3$.

Problem 8. If N is a $k \times k$ nilpotent matrix such that $N^k = 0$ but $N^{k-1} \neq 0$, prove that N is similar to its transpose. (*Hint:* Prove it for a Jordan block matrix N of size k by finding a permutation matrix P with $N^T = PNP^{-1}$)

Problem 9. Prove that two 2×2 matrices over $\mathbb{K}$ are similar if and only if they have the same minimal polynomial.

Problem 10. Prove that two 3×3 matrices over $\mathbb{K}$ are similar if and only if they have the same minimal and characteristic polynomials.

Problem 11. Prove the Cayley-Hamilton Theorem over any commutative ring R: Let $A \in \mathrm{Mat}_{n\times n}(R)$. If $f(X) = \det(X\, I_n - A)$, then $f(A) = 0$.
(*Hint:* Use the identity $\mathrm{Cof}(B)B = B\,\mathrm{Cof}(B) = \det(B)I_n$ for any $n \times n$ matrix B, where $\mathrm{Cof}(B)$ is the cofactor matrix of B).

Problem 12. Let A, B be two $n \times n$ matrices over a field $\mathbb{K}$. Assume that $AB = BA$ and both A and B are diagonalizable (i.e. has a basis of eigenvectors). Show that A and B can be simmultanously diagonalized. (*Hint:* By induction on n. For the inductive step you will need to show that any eigenspace $E_\lambda(A) = \{v \in \mathbb{K}^n : A\,v = \lambda\,v\}$ is invariant under B.)

Problem 13. (Nakayama's Lemma v2)
Consider a finitely generated R-module M and an ideal I of R included in the Jacobson radical of R (see Problem 8 on Homework 7). If $IM = M$, prove that $M = 0$.

Problem 14. (Nakayama's Lemma v3)
Consider a finitely generated R-module M and an ideal I of R. If $M = IM$, then prove that there exists an element $a \in I$ with $m = am$ for all $m \in M$ (equivalently, $(1-a)M = 0$.).

(*Hint for Problems 13 and 14:* Follow the proof of Nakayama's lemma discussed in Lecture 34.)

Problem 15. (Nakayama's Lemma v3 fails over non-commutative rings)
Consider a non-commutative ring A with no zero divisors and let I be a f.g. proper non-zero idempotent ideal (i.e., $I^2 = I$). Show that Nakayama's Lemma v3 from Problem 14 fails for $M = I$.
(*Note:* Examples of such rings arise in Lie Theory. We can take $\mathfrak{g}$ to be a perfect Lie algebra and let $I = \mathfrak{g}A$ be the augmentation ideal of its enveloping algebra $A = U(\mathfrak{g})$)

Problem 16. (An application of Nakayama's lemma due to Vasconcelos)
The goal of this exercise is to extend Problem 15 of Homework 8 to the non-Noetherian case. More precisely, assume M is a finitely generated R-module and let $f : M \to M$ be a surjective R-linear map. We wish to show that f is injective, and hence an isomorphism.

(i) Show that M becomes an $R[x]$ mode via $P(x) \cdot m = P(f)(m)$.

(ii) Show that M is a finitely generated $R[x]$-module with the structure defined in (i).

(iii) Show that ideal $I = (x) \subsetneq R[x]$ satisfies $IM = M$.

(iv) Use Problem 14 to find a polynomial $P(x) \in I$ with $m = P(x)m$ for all $m \in M$.

(v) Use item (iv) to conclude that $\mathrm{Ker}(f) = \{0\}$.

Problem 17. Show that bases of vector spaces of any dimension (defined as linearly independent spanning sets) are maximal linearly independent sets.

Problem 18. Consider a $\mathbb{K}$-vector space V and let $W \subset V$ be a subspace. Show that:

$$(V/W)^* = \{\xi \in V^* : \xi_{|W} = 0\}.$$

Problem 19. Consider a collection $\{V_i : i \in I\}$ of finite-dimensional $\mathbb{K}$-vector spaces and let W be a vector space.

(i) Prove that $\mathrm{Hom}_{\mathbb{K}}(\bigoplus_{i\in I} V_i, W) \simeq \prod_{i\in I} \mathrm{Hom}_{\mathbb{K}}(V_i, W)$.

(ii) Prove that $(\bigoplus_{i\in I} V_i)^* \simeq \prod_{i\in I} (V_i)^*$.

(In both items, direct products/sums of vector spaces are defined as the corresponding operations on $\mathbb{K}$-modules.)

Problem 20. Let V and W be two finite-dimensional $\mathbb{K}$ vector spaces, with bases B_V and B_W, respectively. Assume $\dim V = n$ and $\dim W = m$. Consider any linear transformation $f\colon V \to W$ and its corresponding dual map $f^*\colon W^* \to V^*$.

(i) Show that if $A = [f]_{B_V,B_W}$ is the $m \times n$ matrix representative of f with respect to the bases B_V and B_W, then $A^T = [f^*]_{(B_W)^*,(B_V)^*}$ is the matrix representative of f^* with respect to the dual bases $(B_W)^*$ and $(B_V)^*$.

(ii) Show that $W^* \simeq (\mathrm{Im}\, f)^* \oplus \mathrm{Ker}(f^*)$, where we view $(\mathrm{Im}\, f)^* \subset W^*$ by extending a basis of $\mathrm{Im}\, f$ to a basis of W and taking duals.

(iii) Show that the images of f and f^* have the same dimension, and conclude from this that the (column) ranks of A and A^T agree.

Problem 21. Given a $\mathbb{K}$-vector space V, consider the dual vector space to V^*. We denote it by $(V^*)^*$ and call it the "double dual" of V. Show that we can view V as a subspace of its double dual by writing an explicit injective linear map $V \hookrightarrow (V^*)^*$. If V is finite dimensional, conclude that $V \simeq (V^*)^*$.