ALGEBRA I (MATH 6111 AUTUMN 2021) - HOMEWORK 12

In all problems below, we assume R is a commutative ring and $\mathbb{K}$ is a field.

Problem 1. Show that given two $\mathbb{K}$-vector spaces V_1 and V_2, the tensor product $V_1 \otimes_{\mathbb{K}} V_2$ defined via universal property (see Lecture 36) is unique up to unique isomorphism. Conclude from this that for any $\mathbb{K}$-vector space V, we have

$$V \otimes_{\mathbb{K}} \mathbb{K} \simeq \mathbb{K} \otimes_{\mathbb{K}} V \simeq V.$$

Problem 2. Consider three $\mathbb{K}$-vector spaces V_1, V_2 and W. Show that:

$$(V_1 \oplus V_2) \otimes_{\mathbb{K}} W \simeq (V_1 \otimes_{\mathbb{K}} W) \oplus (V_2 \otimes_{\mathbb{K}} W).$$

Problem 3. Consider two matrices $X_1 \in \mathrm{Mat}_{m_1 \times n_1}(\mathbb{K})$ and $X_2 = \mathrm{Mat}_{m_2 \times n_2}(\mathbb{K})$ representing linear transformations $f_1 \colon V_1 \to W_1$ and $f_2 \colon V_2 \to W_2$ with $\dim V_i = n_i$ and $\dim W_i = m_i$ for $i = 1, 2$. Write down the matrix representing $f_1 \otimes f_2 \colon (V_1 \otimes_{\mathbb{K}} V_2) \to (W_1 \otimes_{\mathbb{K}} W_2)$ using the matrices X_1 and X_2, as in Lecture 36.

Problem 4. Apply the construction from Problem 3 to the following matrices:

$$X_1 = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad X_2 = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Problem 5. Consider two square matrices A_1 and A_2 of sizes $n \times n$ and $m \times m$, respectively. Let $\{\lambda_i : i = 1, \ldots, n\}$ and $\{\mu_j : j = 1, \ldots, m\}$ be the eigenvalues of A_1 and A_2, counted with multiplicity. Show that the eigenvalues of the matrix $A_1 \otimes A_2$ (constructed in Problem 3), counted with multiplicity, are given by $\{\lambda_i \mu_j : i = 1, \ldots, n, j = 1, \ldots m\}$.

Problem 6. Consider three $\mathbb{K}$-vector spaces U, V, W. Assume V is finite-dimensional. Show that $\mathrm{Hom}_{\mathbb{K}}(V, U) \otimes W \simeq \mathrm{Hom}_{\mathbb{K}}(V, U \otimes W)$ by writing an explicit isomorphism. (*Hint:* When $U = \mathbb{K}$, the statement is the Hom-tensor adjointness theorem)

Problem 7. Consider two finite-dimensional $\mathbb{K}$-vector spaces V and W, each with two bases B_1, B_1' and B_2, B_2', respectively. Describe the change of bases matrix for $V \otimes_{\mathbb{K}} W$ with respect to the bases $B_1 \times B_2$ and $B_1' \times B_2'$ (ordered appropriately).

Problem 8. Consider $\mathbb{K}$-vector spaces U, V, W. Using the universal property of tensor products show that

(i) there exists a unique $\mathbb{K}$-linear isomorphism $\phi : V \otimes_{\mathbb{K}} W \xrightarrow{\simeq} W \otimes_{\mathbb{K}} V$ satisfying $\phi(v \otimes w) = w \otimes v$ for all $v \in V$, $w \in W$;

(ii) there exists a unique $\mathbb{K}$-linear isomorphism $\beta : (U \otimes_{\mathbb{K}} V) \otimes_{\mathbb{K}} W \xrightarrow{\simeq} U \otimes_{\mathbb{K}} (V \otimes_{\mathbb{K}} W)$ satisfying $\beta((u \otimes v) \otimes w) = u \otimes (v \otimes w)$.

Problem 9. Consider two finite-dimensional $\mathbb{K}$-vector spaces V and W, with $\dim V = n$ and $\dim W = m$. Consider the following composition of isomorphisms:

$$\alpha : \mathrm{Hom}_{\mathbb{K}}(V^*, W) \xrightarrow{\simeq} V \otimes_{\mathbb{K}} W \xrightarrow{\phi} W \otimes_{\mathbb{K}} V \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{K}}(W^*, V)$$

where the first and the last maps arise from Hom-tensor adjointness, and ϕ is the map defined in Problem 8 (i). Show that under the identifications $\mathrm{Mat}_{m\times n}(\mathbb{K}) \simeq \mathrm{Hom}_{\mathbb{K}}(V^*, W)$ and $\mathrm{Hom}_{\mathbb{K}}(W^*, V) \simeq \mathrm{Mat}_{n\times m}(\mathbb{K})$, the map α corresponds to taking the transpose of a matrix.

Problem 10. (Rank of tensors) Consider two $\mathbb{K}$-vector spaces V and W. For any $u \in V \otimes_{\mathbb{K}} W$ we define the *rank* of u as the smallest non-negative integer r for which u admits an expression:

$$u = \sum_{i=1}^{r} v_i \otimes w_i \qquad v_i \in V, w_i \in W. \tag{1}$$

(i) Assume that $\mathrm{rank}(u) = r$ and write u as in (1). Show that the sets $\{v_i : i = 1, \ldots, r\}$ and $\{w_i : i = 1, \ldots, r\}$ are linearly independent subsets of V and W, respectively.

(ii) Conversely, if $u = \sum_{i=1}^{s} a_i \otimes b_i$ where the sets $\{a_i : i = 1, \ldots, s\} \subset V$ and $\{b_i : i = 1, \ldots, s\} \subset W$ are linearly independent, then $\mathrm{rank}(u) = s$.

(iii) Assume V and W are finite-dimensional. Hom-tensor adjointness yields isomophisms

$$\varphi_1 : V \otimes_{\mathbb{K}} W \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{K}}(V^*, W), \qquad \varphi_2 : W \otimes_{\mathbb{K}} V \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{K}}(W^*, V).$$

Consider the isomorphism ϕ from Problem 8 item (i) and let $u_1 = \varphi_1(u)$, and $u_2 = (\varphi_2 \circ \phi)(u)$. Show that $\mathrm{rank}(u) = \dim \mathrm{Im}(u_1) = \dim \mathrm{Im}(u_2)$.

Problem 11. Let $k \in \mathbb{Z}_{\geq 1}$ and $v_1, \ldots, v_k \in V$. Show that $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k V$ is non-zero if, and only if, $\{v_1, \ldots, v_k\}$ is linearly independent.

Problem 12. Let $P \in \mathrm{End}_{\mathbb{K}}(V)$ satisfying $P^2 = P$ (i.e. P is a projection). Show that $V \simeq \ker(P) \oplus \mathrm{Im}(P)$.

Problem 13. Let S_n be the symmetric group of n letters and V be a $\mathbb{K}$-vector space. Show that we can define a unique left action of S_n on $T^n(V)$ satisfying that $\sigma \cdot (v_1 \otimes \ldots \otimes v_n) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$ for each $\sigma \in S_n$ and each $v_1, \ldots, v_n \in V$. For each $\sigma \in S_n$ we let $\overline{\sigma}\colon T^n(V) \to T^n(V)$ be the unique linear map thus obtained.

Problem 14. ($S^n(V)$ as a subspace of $T^n(V)$)

Assume $\mathrm{char}(\mathbb{K}) = 0$ and consider $S\colon T^n(V) \to T^n(V)$ given by $S(\xi) = \dfrac{1}{n!}\displaystyle\sum_{\sigma \in S_n} \overline{\sigma}(\xi)$, where $\overline{\sigma}(\xi)$ is defined as in Problem 13.

(i) Show that $S^2 = S$ and, furthermore,

$$\ker(S) = \langle v_1 \otimes \ldots v_n - v_1 \otimes \ldots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \ldots \otimes v_n : 1 \leq i \leq n-1, v_1, \ldots, v_n \in V \rangle.$$

(ii) Conclude that $\mathrm{Im}(S) \simeq S^n(V)$ as $\mathbb{K}$-vector spaces.

Problem 15. ($\bigwedge^n(V)$ as a subspace of $T^n(V)$)
Assume $\mathrm{char}(\mathbb{K}) = 0$ and consider $A\colon T^n(V) \to T^n(V)$ given by $A(\xi) = \dfrac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\overline{\sigma}(\xi)$, where $\overline{\sigma}(\xi)$ is defined as in Problem 13.

(i) Show that $A^2 = A$ and furthermore,

$$\ker(A) = \langle v_1\otimes \ldots v_n + v_1\otimes\ldots\otimes v_{i-1}\otimes v_{i+1}\otimes v_i\otimes v_{i+2}\ldots\otimes v_n : 1\leq i\leq n-1, v_1,\ldots,v_n\in V\rangle.$$

(ii) Conclude that $\mathrm{Im}(A) \simeq \bigwedge^n(V)$ as $\mathbb{K}$-vector spaces.

Problem 16. Prove the following isomorphisms of $\mathbb{K}$-vector spaces for all $n \in \mathbb{Z}_{>0}$:

(i) $T^n(V\oplus W) \simeq \bigoplus_{k=0}^{n} \bigoplus_{\substack{i_1\geq 0,\, i_2,\ldots,i_k>0\\ i_1+\ldots i_k=n}} (T^{i_1}(V)\otimes T^{i_2}(W)\otimes T^{i_3}(V)\otimes\ldots)$, (i.e., alternate tensor powers of V and W, allowing to start from V or W, where the sum of the powers equals n)

(ii) $S^n(V\oplus W) \simeq \bigoplus_{i=0}^{n} S^i(V)\otimes S^{n-i}(W)$,

(iii) $\bigwedge^n(V\oplus W) \simeq \bigoplus_{i=0}^{n} \bigwedge^i(V)\otimes\bigwedge^{n-i}(W)$.

Problem 17. Let $v_1,\ldots,v_r \in V$ be a collection of linearly independent vectors. Let $\omega\in\bigwedge^p(V)$. Show that ω can be written as $\omega = \sum_{i=1}^r v_i\wedge\psi_i$ for some $\psi_1,\ldots\psi_r\in\bigwedge^{p-1}(V)$ if, and only if, $v_1\wedge\ldots\wedge v_r\wedge\omega = 0\in\bigwedge^{p+r}(V)$.

Problem 18. Show that the multiplication maps $S^k(V)\times S^\ell(V)\to S^{k+\ell}(V)$ and $\bigwedge^k(V)\times\bigwedge^\ell(V)\to\bigwedge^{k+\ell}(V)$ defined in Lecture 38 are well-defined and are obtained from the multiplication map $\phi\colon T^k(V)\times T^\ell(V)\to T^{k+\ell}(V)$ composed with the corresponding natural projections to the symmetric and exterior powers of V.

Problem 19. Assume $\mathrm{char}(\mathbb{K}) = 0$ and consider the map $\varphi\colon S^k(V)\times S^\ell(V)\to S^{k+\ell}(V)$ defined on the indecomposable tensors via

$$\varphi(\xi,\eta) = \frac{1}{\binom{k+\ell}{k}}\sum_{\sigma\in G}\sigma(\xi\otimes\eta) \qquad \text{for } G = S_{k+\ell}/(S_k\times S_\ell), \xi\in S^k(V), \eta\in S^\ell(V),$$

where we view $S_k\times S_\ell\subset S_{k+\ell}$ as permutations of $\{1,\ldots,k\}$ and $\{k+1,\ldots,k+\ell\}$.

(i) Check that φ is bilinear, so it yields a unique linear map $\overline{\varphi}\colon S^k(V)\otimes S^\ell(V)\to S^{k+\ell}(V)$.

(ii) Show that $\overline{\varphi}$ defines an associative multiplication map on $S^\bullet(V)$.

(iii) Show that $\overline{\varphi}$ fits into the natural commutative diagram involving the multiplication map on $T^\bullet(V)$, the projection $T^n(V)\to S^n(V)$ and the inclusion $S^n(V)\hookrightarrow T^n(V)$

defined in Problem 14:

$$\begin{array}{ccc} S^k(V) \otimes S^\ell(V) & \xrightarrow{\overline{\varphi}} & S^{k+\ell}(V) \\ \big\downarrow \quad\quad \big\downarrow & & \big\uparrow \\ T^k(V) \otimes T^\ell(V) & \xrightarrow{\text{mult.}} & T^{k+\ell}(V) \end{array}$$

Problem 20. Prove that $V \otimes V \simeq S^2(V) \oplus \bigwedge^2(V)$ by writing the explicit isomorphisms.

Problem 21. Assume V is a vector space of dimension n (finite). Pick a basis $B = \{v_1, \ldots, v_n\}$ for V.

(i) Show that $\psi_B \colon \bigwedge^n(V) \to \mathbb{K}$ given by $\psi_B(\alpha(v_1 \wedge \ldots \wedge v_n)) = \alpha$ is an isomorphism.

(ii) If B' is another basis for V, and A is the change of bases matrix from B' to B, show that $\psi_B(\xi) = \det(A)\psi_{B'}(\xi)$ for all $\xi \in \bigwedge^n(V)$.

Problem 22. Assume $\mathrm{char}(\mathbb{K}) = 0$. Consider a $\mathbb{K}$-linear map $f \colon V \to W$ and the associated maps $S^n(f) \colon S^n(V) \to S^n(V)$ and $\bigwedge^n(f) \colon \bigwedge^n(V) \to \bigwedge^n(V)$.

(i) Show that these constructions are compatible with compositions, i.e., if $f \colon V \to U$, and $g \colon U \to W$ are $\mathbb{K}$-linear, then

$$S^n(g \circ f) = S^n(g) \circ S^n(f) \qquad \text{and} \qquad \bigwedge^n(g \circ f) = \bigwedge^n(g) \circ \bigwedge^n(f).$$

(ii) Let $n = \dim(V)$ and pick $f \in \mathrm{End}_{\mathbb{K}}(V)$. Show that the following diagram commutes

$$\begin{array}{ccc} \bigwedge^n(V) & \xrightarrow{\bigwedge^n(f)} & \bigwedge^n(V) \\ \simeq \big\downarrow & & \big\downarrow \simeq \\ \mathbb{K} & \xrightarrow{\det(f)} & \mathbb{K} \end{array} .$$

Here, the bottom map is multiplication by $\det(f) \in \mathbb{K}$, and the vertical maps are any fixed isomorphism from Problem 21.

(iii) In particular, if $V = U = W$ have dimension n and f and g correspond to two matrices $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{K})$, show that $\det(AB) = \det(A)\det(B)$.

Problem 23. Assume $\mathrm{char}(\mathbb{K}) = 0$. Prove the row expansion formula for determinants of square matrices using $\det(f) = \bigwedge^n(f) \colon \bigwedge^n(\mathbb{K}^n) \to \bigwedge^n(\mathbb{K}^n)$ where $f \colon \mathbb{K}^n \to \mathbb{K}^n$ is multiplication by the corresponding matrix.

Problem 24. (Determinants vs. Permanents of singular matrices)
Assume $\mathrm{char}(\mathbb{K}) = 0$ and consider the endomorphism $f \colon \mathbb{K}^2 \to \mathbb{K}^2$ defined by multiplication by the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

(i) Compute $\bigwedge^2(f) \colon \bigwedge^2(\mathbb{K}^2) \to \bigwedge^2(\mathbb{K}^2)$.

(ii) Compute $S^2(f) \colon S^2(\mathbb{K}^2) \to S^2(\mathbb{K}^2)$.

(iii) Compute $S^3(f)\colon S^3(\mathbb{K}^2) \to S^3(\mathbb{K}^2)$.

Problem 25. Let $X \in \mathrm{GL}_n(\mathbb{K})$. We say X admits a *Gaussian decomposition* if it can be written as a product

$$X = X^- \, X^0 \, X^+,$$

where X^0 is a diagonal matrix, X^+ is an upper triangular matrix (i.e., $X^+_{ij} = 0$ for $i > j$) with ones along the diagonal, and X^- is an lower triangular matrix (i.e., $X^-_{ij} = 0$ for $i < j$) with ones along the diagonal.

(i) Show that if X admits a Gaussian decomposition then it is unique. (*Hint:* Prove the uniqueness for diagonal matrices admitting a Gaussian decomposition.)

(ii) Show that a matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{K})$ admits a Gaussian decomposition if, and only if, $a \neq 0$. Compute explicit formulas for X^-, X^0 and X^+.

Problem 26*. Consider $X \in \mathrm{GL}_n(\mathbb{K})$. The goal of this exercise is to show that X admits a Gaussian decomposition if, and only if, all its principal minors are non-zero.

(i) Show that if X admits a Gaussian decomposition, then all principal minors $\Delta^{1,\ldots,i}_{1,\ldots,i}$ (for $i = 1, \ldots, n$) are non-zero.

(ii) If X admits a Gaussian decomposition, show that $X^0_{11} = X_{11}$ and $X^0_{ii} = \Delta^{1,\ldots,i}_{1,\ldots,i} / \Delta^{1,\ldots,i-1}_{1,\ldots,i-1}$ for all $i = 2, \ldots, n$.

(iii) Furthermore, prove that $X^-_{ji} = \Delta^{1,\ldots,i-1,j}_{1,\ldots,i} / \Delta^{1,\ldots,i}_{1,\ldots,i}$ and $X^+_{ij} = \Delta^{1,\ldots,i}_{1,\ldots,i-1,j} / \Delta^{1,\ldots,i}_{1,\ldots,i}$ for all $i \leq j$.

(iv) Conclude that X admits a Gaussian decomposition if, and only if, all its principal minors are non-zero.