Lecture 2: Subgroups, Normal subgps, Hom spaces

Recall: $(G,*,e)$ group
- $a*b \in G \quad \forall a,b\in G$
- e neutral element

(i) Assoc: $(a*b)*c = a*(b*c) \quad \forall a,b,c\in G$

(ii) Neutral: $a*e = e*a = a \quad \forall a\in G$ (unique!)

(iii) Inverse: $\forall a\in G\ \exists b\in G : a*b = b*a = e$ (unique!)

G, G' gps $\quad \varphi: G \longrightarrow G'$ gp homomorphism means $\varphi(a*b) = \varphi(a)*\varphi(b)$ ($\leadsto \varphi(e)=e'$, $\varphi(x^{-1}) = \varphi(x)^{-1}$)

- Nice groups: those given as "symmetries of a structure"
  Advantage: Associativity is automatic!

"Structure": a finite set $X = \{1,2,\dots,n\}$ for $n=|X|$

"Symmetries" = bijections $\sigma: X \longrightarrow X$

Group operation = composition of two maps

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[rr, bend right] & X \arrow[r, "\tau"] & X \end{tikzcd}$$

$\tau * \sigma := \tau\circ\sigma = \tau\sigma$ (usually we omit $\circ$)

Ex: S_n = permutations on n letters ; D_n = symmetries of n-gon

- More examples: $\mathbb{F}_n$ = free group on n letters $\{a_1,\dots,a_n\}$

$\mathbb{F}_n$ = "words" in the alphabet (e = "empty word")

$*$ = concatenation (+ cancellations)

Eg: $a_2^{-2}a_1a_2a_1 * a_1^{-1}a_2 = a_2^{-2}a_1a_2\underbrace{\cancel{a_1}\cancel{a_1^{-1}}}a_2 = a_2^{-2}a_1a_2^2$

$(a_1^{-1}a_2)^{-1} = a_2^{-1}a_1$, $(a_2^{-2}a_1a_2a_1)^{-1} = a_1^{-1}a_2^{-1}a_1^{-1}a_2^2$

§1 Subgroups: G gp

Def A subset $H \subset G$ is a subgroup of G if:

(i) $e \in H$

(ii) $x, y \in H \Rightarrow x * y \in H$

(iii) $x \in H \Rightarrow x^{-1} \in H$

(H inherits gp structure from G)

Notation: $H < G$ for subgroup.

Obs: (ii) + (iii) can be written together as $\boxed{x, y \in H \Rightarrow x y^{-1} \in H}$

Proposition A nonempty subset H of G is a subgroup if, and only if, for all $x, y \in H$ we have $x * y^{-1} \in H$.

Pf/ ($\Rightarrow$) If H is a subgroup and $y \in H$, we have $y^{-1} \in H$ as well. Since H is closed under $*$ & $x, y^{-1} \in H$, we conclude $x * y^{-1} \in H$

($\Leftarrow$) We must show properties (i), (ii) & (iii)

(i) Since $H \neq \emptyset$ we can pick an $x \in H$. Then take $y = x$ & conclude that $e = x * x^{-1} \in H$

(iii) Pick any $y \in H$. Since $e \in H$ by (i) we conclude $y^{-1} = e * y^{-1} \in H$. So H is close under taking inverses, as we want to show.

(ii) Pick any $x, y \in H$. Since $y \in H$, by (iii) we have $y^{-1} \in H$. Therefore $x, y^{-1} \in H$ forces $x * y = x * (y^{-1})^{-1} \in H$ by our hypothesis. We conclude that H is closed under the operation $*$ □

§2 Normal subgroups:

Def: A subgroup $H < G$ is called normal if $\forall a \in G$, $b \in H$ we have $a^{-1} b a \in H$ (equivalently $a b a^{-1} \in H$)

Notation: $H \triangleleft G$ (\triangleleft in LaTeX)

Obs: If G is abelian, every subgroup H of G is normal.

(Is the converse true? $\rightsquigarrow$ see page 9)

Q: Subgroups from gp homomorphisms? **A:** Yes (just as in Linear Algebra)

Def: Given $\varphi : G \longrightarrow G'$ gp homomorphism

$$\mathrm{Ker}(\varphi) := \{ x \in G : \varphi(x) = e' \} = \text{Kernel of } \varphi$$

$$\mathrm{Im}(\varphi) := \{ \varphi(x) : x \in G \} = \text{Image of } \varphi$$

Lemma: (1) $\mathrm{Ker}(\varphi) \lhd G$ & (2) $\mathrm{Im}(\varphi) < G'$

Pf/ (1) Claim 1: $\mathrm{Ker}(\varphi) < G$

Need to check 3 properties defining subgroups. Alternatively, we use the Proposition from page 2. This requires us to check 2 things:

- $\mathrm{Ker}\,\varphi \neq \emptyset$: It holds since $\varphi(e) = e'$ says $e \in \mathrm{Ker}\,\varphi$.
  (Lecture 1)
- We must show: $x, y \in \mathrm{Ker}\,\varphi \overset{?}{\Longrightarrow} x * y^{-1} \in \mathrm{Ker}\,\varphi$.

Again, we use the fact that φ is a group homomorphism.

$$\varphi(x * y^{-1}) = \varphi(x) * \varphi(y^{-1}) = \varphi(x) * \varphi(y)^{-1} \underset{x,y \in \mathrm{Ker}\,\varphi}{=} e' * (e')^{-1} = e' * e' = e'.$$

Claim 2: $b \in \mathrm{Ker}(\varphi),\ a \in G \Rightarrow a^{-1} b a \in \mathrm{Ker}\,\varphi$.

Indeed: $\varphi(a^{-1} b a) \underset{b \in \mathrm{Ker}\,\varphi}{=} \varphi(a)^{-1} \underbrace{\varphi(b)}_{=e'} \varphi(a) = e'$ ✓

(2) $\mathrm{Im}(\varphi) < G'$ $\rightsquigarrow$ exercise $\square$

⚠ $\mathrm{Im}\,\varphi$ need not be normal

Ex: $G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : \begin{array}{c} a,b,c \in \mathbb{C} \\ a,c \neq 0 \end{array} \right\} \xhookrightarrow[\varphi]{} GL_2(\mathbb{C})$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \underset{\varphi(G)}{\underset{\cap}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \notin \varphi(G)$$

<u>Obs</u>: If G is abelian, then all its subgroups are normal. The converse is not true: the quaternions will provide an example (see HW1)

§3. Hom Spaces:

- $\mathrm{Hom}_{Gps}(G, G')$ = Set of group homomorphisms $G \to G'$
- $\varphi \in \mathrm{Hom}(G, G')$ is an <u>isomorphism</u> if $\exists\ \varphi' \in \mathrm{Hom}(G', G)$ st
$$\varphi \circ \varphi' = id_{G'}$$
$$\varphi' \circ \varphi = id_G$$

<u>Def</u>: G & G' are <u>isomorphic</u> groups (write $G \simeq G'$) if $\exists\ \varphi \in \mathrm{Hom}(G, G')$ isomorphism.

- We define the following operation on $\mathrm{Hom}_{Gps}(G, G')$
$$\varphi, \varphi' \in \mathrm{Hom}_{Gps}(G, G') \rightsquigarrow \varphi * \varphi' : G \longrightarrow G'$$
$$g \longmapsto \varphi(g) * \varphi'(g)$$
(This is pointwise evaluation)

This operation defines a map $\mathrm{Hom}_{Gps}(G, G') \times \mathrm{Hom}_{Gps}(G, G') \to \mathrm{Hom}_{Sets}(G, G')$
- Associative because $*_{G'}$ is associative
- Neutral element $\varphi' : G \to G'$ $\varphi(g') = e'$.
- Inverses?: $\varphi'(g) = (\varphi(g))^{-1} : G \longrightarrow G'$ is well-defined but it's not necessarily a group homomorphism.
$$\varphi'(gh) = (\varphi(gh))^{-1} = (\varphi(g)\varphi(h))^{-1} = \varphi(h)^{-1}\varphi(g)^{-1}$$
$$= \varphi'(h)\varphi'(g)$$

So, to get $\varphi'(gh) = \varphi'(g)\varphi'(h)$ we need $\varphi'(h)$ & $\varphi'(g)$ to commute! This will be true if G' is abelian.

This is not the only issue we face:

Prop: $\varphi * \varphi'$ is a group homomorphism if G' is abelian

Pf/ $(\varphi * \varphi')_{(g*h)} = \varphi_{(g*h)} * \varphi'_{(g*h)} = (\varphi_{(g)} * \varphi_{(h)}) * (\varphi'_{(g)} * \varphi'_{(h)})$

$= \varphi_{(g)} * (\varphi_{(h)} * \varphi'_{(g)}) * \varphi'_{(h)}$

Want $(\varphi * \varphi')_{(g*h)} = (\varphi * \varphi')_{(g)} * (\varphi * \varphi')_{(h)} = (\varphi_{(g)} * \varphi'_{(g)}) * (\varphi_{(h)} * \varphi'_{(h)})$

$= \varphi(g) * (\varphi'_{(g)} * \varphi_{(h)}) * \varphi'(h)$

Conclusion: $(\varphi * \varphi')_{(g*h)} = (\varphi * \varphi')_{(g)} * (\varphi * \varphi')_{(h)}$ if, and only if,

$\varphi(h) * \varphi'(g) = \varphi'(g) * \varphi(h)$

This will hold if G' is abelian. $\square$

Conclusion: Our calculations say $\mathrm{Hom}_{Set}(G, G')$ is a group under pointwise evaluation, but $\mathrm{Hom}_{Gp}(G, G')$ is not necessarily a subgroup for general G'. It's not even a monoid if G' is arbitrary.

If G' is abelian, we can view $\mathrm{Hom}_{Gp}(G, G')$ as a group under pointwise evaluation.

Q: What if $G' = G$?

- $\mathrm{End}(G) = \mathrm{Hom}_{Gp}(G, G)$ is a monoid under compositions
- $\mathrm{Aut}(G)$ = isomorphisms in $\mathrm{Hom}_{Gp}(G, G)$ is a group

§3 Aside: The Quaternion group:

Def: The Quaternion group Q_8 has group presentation

$$Q_8 = \langle \bar{e}, i, j, k \mid \bar{e}^2 = e,\ i^2 = j^2 = k^2 = ijk = \bar{e} \rangle$$

Explicitly: Write $e = 1$ & $\bar{e} = -1$ Then:

$$\boxed{Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}}$$

How to read this from the presentation?

- $-i := \bar{e}\, i = i\,\bar{e}$ ($\bar{e}$ & i commute since $i^2 = \bar{e}$)
- $(i)^{-1} = -i$ because $i^2 = -1$.
- $ij = k$ since $ijk = k^2 \Rightarrow ijkk^{-1} = k^2k^{-1}$.

$\rightsquigarrow$ Cayley Table (multiplication Table) is:

	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	i	j	k	-1	$-i$	$-j$	$-k$
i	i	-1	k	$-j$	$-i$	1	$-k$	j
j	j	$-k$	-1	i	$-j$	k	1	$-i$
k	k	j	$-i$	-1	$-k$	$-j$	i	1
-1	-1	$-i$	$-j$	$-k$	1	i	j	k
$-i$	$-i$	1	$-k$	j	i	-1	k	$-j$
$-j$	$-j$	k	1	$-i$	j	$-k$	-1	i
$-k$	$-k$	$-j$	i	1	k	j	$-i$	-1

Each entry $(\ )_{xy} = xy$

Obs: Q_8 is non-abelian

$ij = k \qquad ji = -k$

& $k \neq -k$

Obs 2: Proper Subgroups of Q_8 are $\{\pm 1\}, \langle i \rangle, \langle j \rangle, \langle k \rangle$

(Idea: if you have two symbols, say i, j, then you generate all of Q_8)

$\langle i \rangle = \{\pm 1, \pm i\}$, etc.

Obs 3: These subgroups are normal

Pf/ ± 1 commutes with all elements

$$g\langle i\rangle g^{-1} = \langle g i g^{-1}\rangle$$

- $g = j \rightsquigarrow j i j^{-1} = j i(-j) = (-k)(-j) = -i \in \langle i\rangle$

$g = k \rightsquigarrow k i k^{-1} = k i(-k) = j(-k) = i \in \langle i\rangle$

Others follow from this because -1 is <u>central</u> (commutes with all other elements)

and $-g\langle i\rangle(-g)^{-1} = g\langle i\rangle g^{-1}$.

<u>Conclusion</u>: Q_8 is NOT abelian & all its subgroups are normal.

($\rightsquigarrow$ example of a Hamiltonian gp)